"Inference of coupling of waves in non linear disordered medium."

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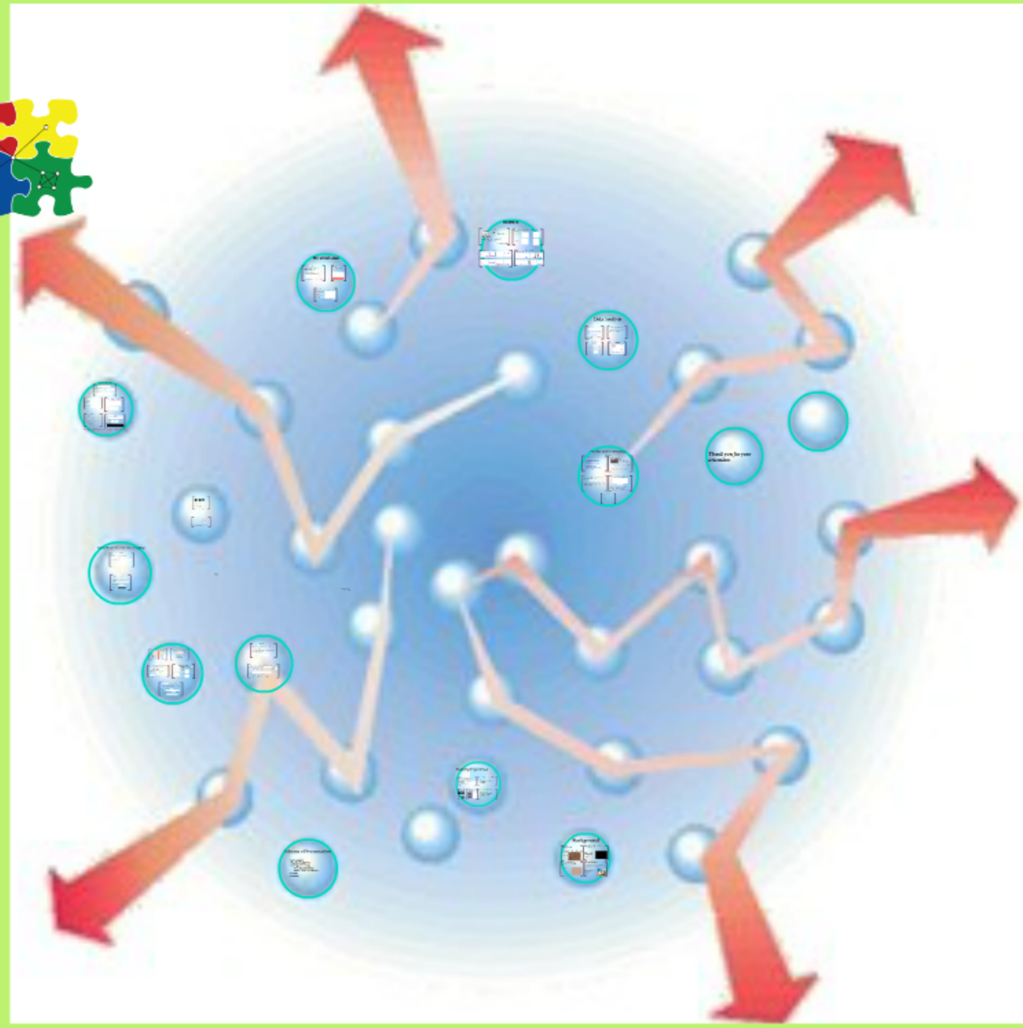

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MARIE CURIE ACTIONS

SEVENTH FRAMEWORK PROGRAMME

IPCF

Consiglio Nazionale delle Ricerche

SAPIENZA UNIVERSITÀ DI ROMA

Supervisor :
Dr. Luca Leuzzi

ESR :
Payal Tyagi

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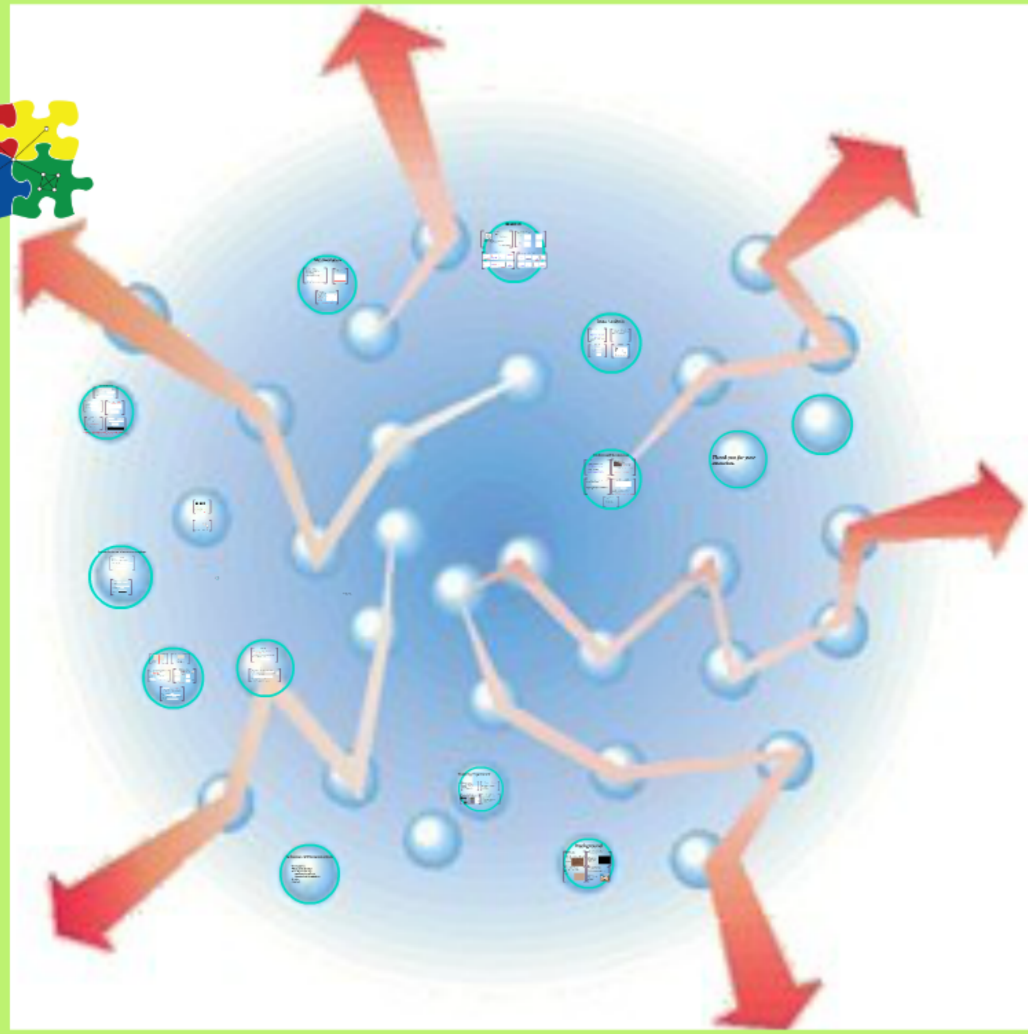

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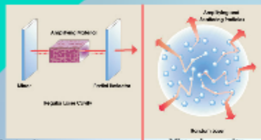
Scheme of Presentation

- *Introduction*
- *Physical motivation*
- *Investigation using :*
  - *- Statistical inference*
  - *- Monte Carlo simulations*
- *Results*
- *Outlook*

Waves in (non-) random media



- Fabry Perot cavity
- Ordered & homogeneous
- Amplifying medium
- Stimulated emssion
  --> multimode lasing above threshold

- Mirror less cavity
- random medium
- Multiple scattering
- Random Lasing (above threshold)

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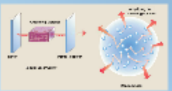

- Below threshold :
  - radiative (extended) modes
  - linear interaction dominates : CW
  - incoherent regime
- Above threshold
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Haus Master equation for standard multimode in ordered(closed) cavity:

$$\dot{a}_n(t)=\underbrace{(g_n-l_n+iD_n)}_{\text{Linear contribution}}a_n+\underbrace{(\gamma-i\delta)}_{\text{Non-linear contribution}}\Sigma_{\omega_j+\omega_k=\omega_l+\omega_n}a_j^*a_k a_l+\eta_n(t)$$

which for disordered cavity takes for of 2 + 4 body Hamiltonian :

$$\mathcal{H} = -\mathcal{R}[\sum_{(ij)} G_{ij}^{(2)} a_i a_j^* + \sum_{\omega_i+\omega_j=\omega_k+\omega_l} G_{ijkl}^{(4)} a_i a_j^* a_k a_l^*]$$

Pairwise interaction coupling physical interpretation:

$$G_{ij}^{(2)} = G_{ii}\delta_{ij} + G_{ij}^{rad} + G_{ij}^{ioh}$$

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L. Angelani, C. Conti, G. Ruocco, and F. Zamponi,Phys. Rev. B 74, 104207 (2006)
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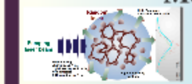
Mode Localization



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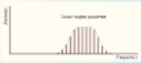

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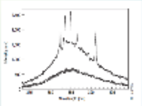
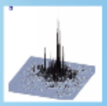


CW --> RL

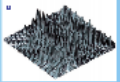
Essential ingredients :
- *Multiple scattering + Gain in the medium*

--> Lasing
--> Localization of modes

Strong localization



Weak Localization



(Radiative modes are relevant)

HC et al, Phys. Rev. Lett. 82, 2278 (1999)

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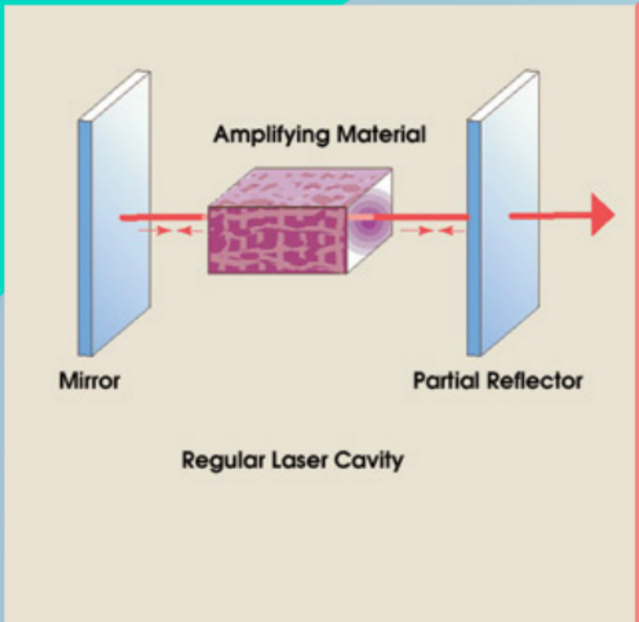
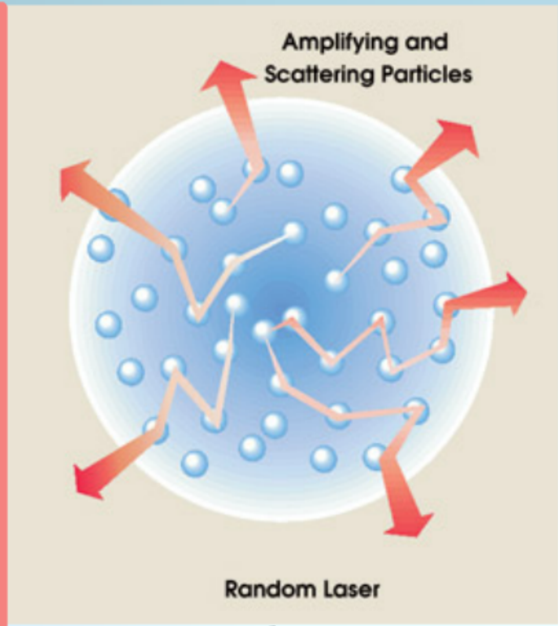
Waves in (non-) random media





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Prezi

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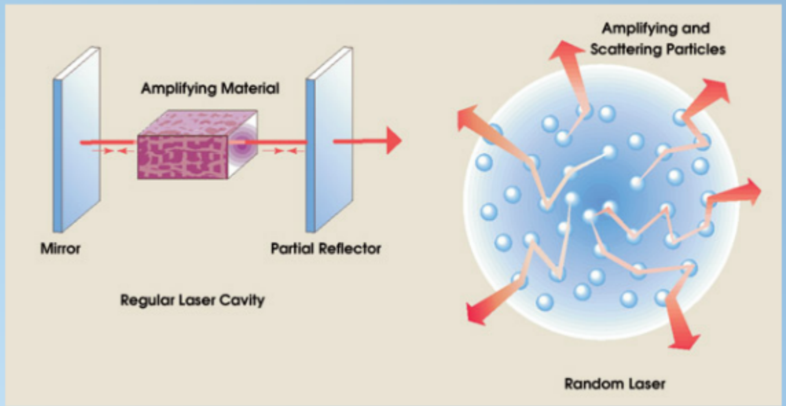

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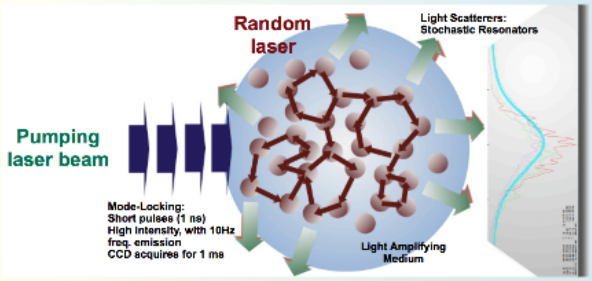

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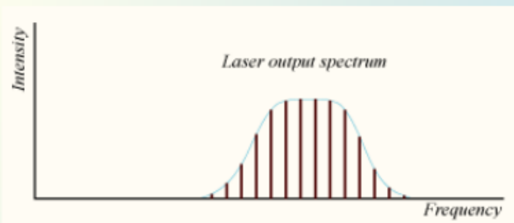

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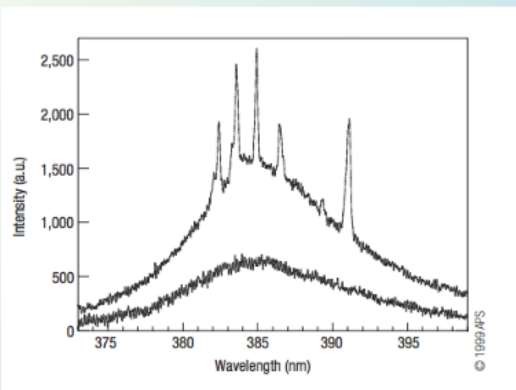

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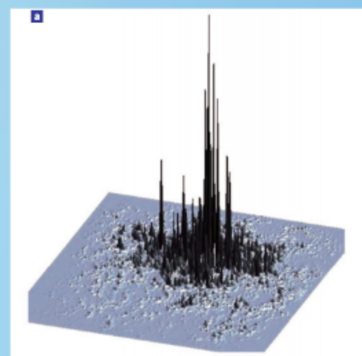

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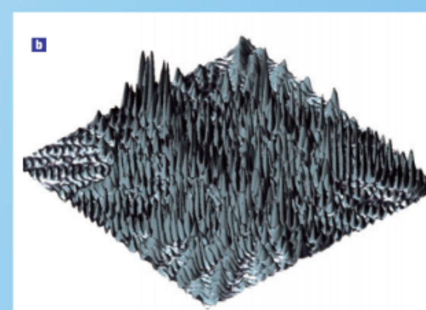

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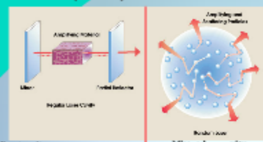

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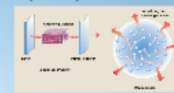

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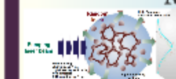

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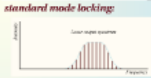

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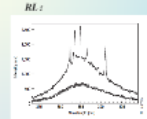

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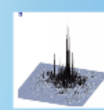

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Useful?

Yes.

Such a Hamiltonian of pairwise interaction can be used for investigating "open" cavities like RL.

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- Also, in **Kuramoto Model** for synchronisation probelms

$$H = -\sum_{ij} J_{ij}\cos(\phi_i - \phi_j) - \sum_i \omega_i \phi_i$$

Hamiltonian

We confine ourselves to the incoherent regime of **linear interaction** among wave in random media.

Final form of Hamiltonian concerning linear interaction and only off diagonal terms:

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Statistical inference of interaction couplings

XY model

Hamiltonian : $H = -\sum_{i<j} J_{ij}\cos(\phi_i-\phi_j) - \sum_i h_i\cos(\phi_i)$

In the mean field approximation applying a variational free energy approach:

$$F = E - S - \sum_i \lambda_i \left(\int_0^{2\pi} d\phi_i \rho(\phi_i) - 1\right) \qquad \frac{\delta F}{\delta\rho(\phi_i)} = 0$$

Probability distribution function : $\rho(\phi_i) = \frac{\exp[R_i\cos(\phi_i-\alpha_i)]}{I_0(R_i)}$ $\qquad Z = \int_0^{2\pi} d\phi_i \exp[R_i\cos(\phi_i-\alpha_i)] = I_0(R_i)$

with :

$$A_i = \sum_j J_{ij}\langle\cos(\phi_j)\rangle + h_i \ , \quad B_i = \sum_j J_{ij}\langle\sin(\phi_j)\rangle$$

$$R_i = \sqrt{A_i^2 + B_i^2} \ , \quad \alpha_i = \arctan\frac{B_i}{A_i} \qquad A_i = R_i\cos\alpha_i \ , \quad B_i = R_i\sin\alpha_i$$

$$I_0(x) = \int_0^{2\pi} d\phi\, e^{x\cos(\phi)} \ , \qquad I_0'(x) = I_1(x)$$

$$I_1(x) = \int_0^{2\pi} d\phi\cos(\phi) e^{x\cos(\phi)} \ , \qquad I_1'(x) = I_0(x) - \frac{I_1(x)}{x}$$

Magnetizations :

$m_i^x = \langle\cos(\phi_i)\rangle =$ [illegible] $\qquad m_i^y = \langle\sin(\phi_i)\rangle =$ [illegible]

Special Case : h =0

$$A_i = \sum_j J_{ij} m_j^x \ , \quad B_i = 0 \ , \quad \alpha_i = 0 \ , \quad R_i = A_i \ , \quad m_i^y = 0 \ , \quad m_i^x = \frac{I_1(R_i)}{I_0(R_i)}$$

Correlation function : $C_{ik} = \frac{\delta m_i^x}{\delta h_k} = \frac{\delta m_i^x}{\delta R_i}\frac{\delta R_i}{\delta h_k}$

$$C_{ik} = [\langle\cos^2(\phi_i)\rangle - \langle\cos(\phi_i)\rangle^2]\left[\sum_j J_{ij} C_{jk} + \delta_{ik}\right]$$

Interaction coupling : $J_{ik} = \frac{\delta_{ik}}{[\langle\cos^2(\phi_i)\rangle - \langle\cos(\phi_i)\rangle^2]} - C_{ik}^{-1}$

XY model

Hamiltonian :

$$H = -\sum_{i<j} J_{ij}\cos(\phi_i - \phi_j) - \sum_i h_i \cos(\phi_i)$$

In the mean field approximation applying a variational free energy approach:

$$F = E - S - \sum_i \lambda_i \left(\int_0^{2\pi} d\phi_i \rho(\phi_i) - 1\right)$$

$$\frac{\delta F}{\delta \rho(\phi_i)} = 0$$

Probability distribution function :

$$\rho(\phi_i) = \frac{\exp\left(R_i \cos(\phi_i - \alpha_i)\right)}{I_0(R_i)}$$

$$Z = \int_0^{2\pi} d\phi_i \exp\left(R_i \cos(\phi_i - \alpha_i)\right) = I_0(R_i)$$

with :

$$A_i = \sum_j J_{ij}\langle\cos(\phi_j)\rangle + h_i \quad , \quad B_i = \sum_j J_{ij}\langle\sin(\phi_j)\rangle$$

$$R_i = \sqrt{(A_i^2 + B_i^2)} \quad , \quad \alpha_i = \arctan\frac{B_i}{A_i}$$

$$A_i = R_i \cos\alpha_i \quad , \quad B_i = R_i \sin\alpha_i$$

$$I_0(z) = \int_0^{2\pi} d\phi e^{(z\cos(\phi))} \quad , \quad I_0^{'}(z) = I_1(z)$$

$$I_1(z) = \int_0^{2\pi} d\phi \cos(\phi) e^{(z\cos(\phi))} \quad , \quad I_1^{'}(z) = I_0(z) - \frac{I_1(z)}{z}$$

Magnetizations :

$$m_i^x = \langle \cos(\phi_i) \rangle = \frac{I_1 \cos(\alpha_i)}{I_0}$$

$$m_i^y = \langle \sin(\phi_i) \rangle = \frac{I_1 \sin(\alpha_i)}{I_0}$$

Special Case : h =0

$$A_i = \sum_j J_{ij} m_j^x \quad , \quad B_i = 0 \quad , \quad \alpha_i = 0 \quad , \quad R_i = A_i \quad , \quad m_i^y = 0 \quad , \quad m_i^x = \frac{I_1(R_i)}{I_0(R_i)}$$

Correlation function :

$$C_{ik}^x = \frac{\delta m_i^x}{\delta h_k} = \frac{\delta m_i^x}{\delta R_i} \frac{\delta R_i}{\delta h_k}$$

$$C_{ik}^x = [\langle \cos^2(\phi_i) \rangle - \langle \cos(\phi_i) \rangle^2][\sum_j J_{ij} C_{jk}^x + \delta_{ik}]$$

Interaction coupling :

$$J_{il} = \frac{\delta_{il}}{[\langle \cos^2(\phi_i) \rangle - \langle \cos(\phi_i) \rangle^2]} - C_{il}^{-1}$$

Prezi

Statistical inference of interaction couplings

XY model

Hamiltonian : $H = -\sum_{i<j} J_{ij}\cos(\phi_i - \phi_j) - \sum_i h_i \cos(\phi_i)$

In the mean field approximation applying a variational free energy approach:

$$F = E - S - \sum_i \lambda_i \left(\int_0^{2\pi} d\phi_i \rho(\phi_i) - 1 \right) \qquad \frac{\delta F}{\delta \rho(\phi_i)} = 0$$

Probability distribution function : $\rho(\phi_i) = \frac{\exp[R_i \cos(\phi_i - \alpha_i)]}{I_0(R_i)}$ $\qquad Z = \int_0^{2\pi} d\phi_i \exp[R_i \cos(\phi_i - \alpha_i)] = I_0(R_i)$

with :

$$A_i = \sum_j J_{ij} \langle \cos(\phi_j) \rangle + h_i \ , \quad B_i = \sum_j J_{ij} \langle \sin(\phi_j) \rangle$$

$$R_i = \sqrt{(A_i^2 + B_i^2)} \ , \quad \alpha_i = \arctan \frac{B_i}{A_i} \qquad A_i = R_i \cos\alpha_i \ , \quad B_i = R_i \sin\alpha_i$$

$$I_0(x) = \int_0^{2\pi} d\phi e^{x\cos(\phi)} \ , \qquad I_0'(x) = I_1(x)$$

$$I_1(x) = \int_0^{2\pi} d\phi \cos(\phi) e^{x\cos(\phi)} \ , \qquad I_1'(x) = I_0(x) - \frac{I_1(x)}{x}$$

Magnetizations :

$$m_i^x = \langle \cos(\phi_i) \rangle = \frac{I_1(R_i) A_i}{I_0(R_i) R_i} \qquad m_i^y = \langle \sin(\phi_i) \rangle = \frac{I_1(R_i) B_i}{I_0(R_i) R_i}$$

Special Case : h =0

$$A_i = \sum_j J_{ij} m_j^x \ , \quad B_i = 0 \ , \quad \alpha_i = 0 \ , \quad R_i = A_i \ , \quad m_i^y = 0 \ , \quad m_i^x = \frac{I_1(R_i)}{I_0(R_i)}$$

Correlation function : $C_{ik} = \frac{\delta m_i^x}{\delta h_k} = \frac{\delta m_i^x}{\delta R_i} \frac{\delta R_i}{\delta h_k}$

$$C_{ik} = [\langle \cos^2(\phi_i) \rangle - \langle \cos(\phi_i) \rangle^2] \left[\sum_j J_{ij} C_{jk} + \delta_{ik} \right]$$

Interaction coupling :

$$J_{ik} = \frac{\delta_{ik}}{[\langle \cos^2(\phi_i) \rangle - \langle \cos(\phi_i) \rangle^2]} - C_{ik}^{-1}$$

Generic XY model

Hamiltonian :

$$\mathcal{H} = -\sum_{(ij)}[J^R_{ij}\cos(\phi_i-\phi_j) - J^I_{ij}\sin(\phi_i-\phi_j)] - \sum_i [h^R_i\cos(\phi_i) + h^I_i\sin(\phi_i)]$$

Applying the same approach :

$$\rho(\phi_i) = \frac{\exp\left(R_i\cos(\phi_i-\alpha_i)\right)}{I_0(R_i)}$$

$$A_i = \sum_j J^R_{ij}\langle\cos(\phi_j)\rangle - \sum_j J^I_{ij}\langle\sin(\phi_j)\rangle + h^R_i$$
$$B_i = \sum_j J^R_{ij}\langle\sin(\phi_j)\rangle + \sum_j J^I_{ij}\langle\cos(\phi_j)\rangle + h^I_i$$

$$R_i = \sqrt{(A_i^2+B_i^2)} \quad , \quad \alpha_i = \arctan\frac{B_i}{A_i}$$

Correlation functions

$$C^X_{ik} = \frac{\delta m^x_i}{\delta h^R_k} \quad , \quad \tilde{C}^X_{ik} = \frac{\delta m^x_i}{\delta h^I_k} \quad , \quad C^Y_{ik} = \frac{\delta m^y_i}{\delta h^R_k}; \quad \tilde{C}^Y_{ik} = \frac{\delta m^y_i}{\delta h^I_k}.$$

$$q_i = \langle\cos^2(\phi_i)\rangle \quad , \quad \frac{I'_1}{I_0} - \frac{I_1^2}{I_0^2} = q_i - |m_i|^2 \quad , \quad |m_i|^2 = (m^x_i)^2 + (m^y_i)^2 \quad , \quad \mu_i = \frac{m^y_i}{m^x_i}$$

$$\frac{A_i}{R_i} = \cos\alpha_i = \cos[\arctan(\mu_i)] = \sqrt{\frac{1}{1+\mu_i^2}}$$
$$\frac{B_i}{R_i} = \sin\alpha_i = \sin[\arctan(\mu_i)] = \sqrt{\frac{\mu_i}{1+\mu_i^2}}$$

Magnetizations :

$$m^x_i = \frac{I_1(R_i)}{I_0(R_i)}\frac{A_i}{R_i} \quad ; \quad m^y_i = \frac{I_1(R_i)}{I_0(R_i)}\frac{B_i}{R_i}$$

Correlation functions :

$$C^X_{ik} = \frac{\delta m^x_i}{\delta h_k} = \frac{A_i}{R_i}\left(\frac{I'_1}{I_0} - \frac{I_1^2}{I_0^2}\right)\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta A_i}{\delta h_k} - \frac{A_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

$$C^Y_{ik} = \frac{\delta m^y_i}{\delta h_k} = \frac{B_i}{R_i}[q - |m|^2]\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta B_i}{\delta h_k} - \frac{B_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

Correlation functions :

$$C^X_{ik} = f_1^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}C^X_{jk} - \sum_j J^I_{ij}C^Y_{jk} - \delta_{ik}\right] + g^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}C^Y_{jk} - \sum_j J^I_{ij}C^X_{jk}\right]$$
$$\tilde{C}^X_{ik} = f_1^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}\tilde{C}^X_{jk} - \sum_j J^I_{ij}\tilde{C}^Y_{jk}\right] + g^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}\tilde{C}^Y_{jk} + \sum_j J^I_{ij}\tilde{C}^X_{jk} + \delta_{ik}\right]$$
$$C^Y_{ik} = g^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}C^X_{jk} - \sum_j J^I_{ij}C^Y_{jk} + \delta_{ik}\right] + f_2^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}C^Y_{jk} - \sum_j J^I_{ij}C^X_{jk}\right]$$
$$\tilde{C}^Y_{ik} = g^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}\tilde{C}^X_{jk} - \sum_j J^I_{ij}\tilde{C}^Y_{jk}\right] + f_2^{(i)}(q_i,m_i)\left[\sum_j J^R_{ij}\tilde{C}^Y_{jk} + \sum_j J^I_{ij}\tilde{C}^X_{jk} - \delta_{ik}\right]$$

$$f_1^{(i)}(q_i,m_i) = \frac{q_i - |m_i|^2 + \mu_i^2 - q_i\mu_i^2}{1+\mu_i^2}$$
$$f_2^{(i)}(q_i,m_i) = \frac{1 + \mu_i + \mu_i^2 - q_i - \mu_i|m_i|^2 - q_i\mu_i^2}{1+\mu_i^2}$$
$$g^{(i)}(q_i,m_i) = \frac{2\sqrt{\mu_i}q_i - \sqrt{\mu_i}|m_i|^2 - \sqrt{\mu_i}}{1+\mu_i^2}$$

What's next?

- Numerical test
- We need data :
- Magnetizations, correlation functions
  --> estimate J

- XY model
- Critical behavior of the XY model in complex topologies , Miguel Ibáñez Berganza, Luca Leuzzi, Phys. Rev. B 88, 144104 (2013)

- Generic XY model

  -- > Set up Monte Carlo simulations.

$$C^x_{ik} = f_1^{(i)}\Gamma^A_{ik} + g^{(i)}\Gamma^B_{ik}$$
$$C^y_{ik} = g^{(i)}\Gamma^A_{ik} + f_2^{(i)}\Gamma^B_{ik}$$

→

$$\Gamma^A_{ik} = k_1^{(i)}C^x_{ik} - k_2^{(i)}C^y_{ik}$$
$$\Gamma^B_{ik} = k_3^{(i)}C^x_{ik} - k_4^{(i)}C^y_{ik}$$

$$k_1^{(i)} = \frac{f_2}{f_1 f_2 - g^2}, \quad k_2^{(i)} = \frac{g}{f_1 f_2 - g^2}, \quad k_3^{(i)} = \frac{g}{g^2 - f_1 f_2}, \quad k_4^{(i)} = \frac{f_1}{g^2 - f_1 f_2}$$

Next, solving :

$$\sum_j J^R_{ij}C^x_{jk} - \sum_j J^I_{ij}C^y_{jk} + \delta_{ik} = \Gamma^A_{ik}$$
$$\sum_j J^R_{ij}C^y_{jk} + \sum_j J^I_{ij}C^x_{jk} = \Gamma^B_{ik}$$

Interaction coupling :

$$J^R_{il} = \sum_k[(\Gamma^A_{ik} - \delta_{ik})(C^y_{kl})^{-1} + \Gamma^B_{ik}(C^x_{kl})^{-1}]\left[\sum_k(C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})\right]^{-1}$$
$$J^I_{il} = \sum_k[-(\Gamma^A_{ik} - \delta_{ik})(C^x_{kl})^{-1} + \Gamma^B_{ik}(C^y_{kl})^{-1}]\left[\sum_k(C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})\right]^{-1}$$

In Collaboration with Dr. Andrea Pagnani(HuGeF Torino)

Generic XY model

Hamiltonian :

$$\mathcal{H} = -\sum_{(ij)}[J^R_{ij}\cos(\phi_i-\phi_j)+J^I_{ij}\sin(\phi_i-\phi_j)] - \sum_i [h^R_i\cos(\phi_i)+h^I_i\sin(\phi_i)]$$

Applying the same approach :

$$\rho(\phi_i) = \frac{\exp\left(R_i\cos(\phi_i-\alpha_i)\right)}{I_0(R_i)}$$

$$A_i = \sum_j J^R_{ij}\langle\cos(\phi_j)\rangle - \sum_j J^I_{ij}\langle\sin(\phi_j)\rangle + h^R_i$$

$$B_i = \sum_j J^R_{ij}\langle\sin(\phi_j)\rangle + \sum_j J^I_{ij}\langle\cos(\phi_j)\rangle + h^I_i$$

$$R_i = \sqrt{(A_i^2+B_i^2)} \quad , \quad \alpha_i = \arctan\frac{B_i}{A_i}$$

Correlation functions

$$C^X_{ik} = \frac{\delta m^x_i}{\delta h^R_k} \quad , \quad \tilde{C}^X_{ik} = \frac{\delta m^x_i}{\delta h^I_k} \quad , \quad C^Y_{ik} = \frac{\delta m^y_i}{\delta h^R_k} , \quad \tilde{C}^Y_{ik} = \frac{\delta m^y_i}{\delta h^I_k} .$$

$- |m_i|^2 \quad , \quad |m_i|^2 = (m_i^x)^2 + (m_i^y)^2 \quad , \quad \mu_i = \frac{m_i^y}{\dots}$

Correlation functions :

$$q_i = \langle \cos^2(\phi_i) \rangle \quad , \quad \frac{I_1'}{I_0} - \frac{I_1^2}{I_0^2} = q_i - |m_i|^2 \quad , \quad |m_i|^2 = (m_i^x)^2 + (m_i^y)^2 \quad , \quad \mu_i = \frac{m_i^y}{m_i^x}$$

$$\frac{A_i}{R_i} = \cos\alpha_i = \cos[\arctan(\mu_i)] = \sqrt{\frac{1}{1+\mu_i^2}}$$

$$\frac{B_i}{R_i} = \sin\alpha_i = \sin[\arctan(\mu_i)] = \sqrt{\frac{\mu_i}{1+\mu_i^2}}$$

Magnetizations :

$$m_i^x = \frac{I_1(R_i)}{I_0(R_i)} \frac{A_i}{R_i} \quad , \quad m_i^y = \frac{I_1(R_i)}{I_0(R_i)} \frac{B_i}{R_i}$$

Correlation functions :

$$C_{ik}^X = \frac{\delta m_i^x}{\delta h_k} = \frac{A_i}{R_i}\left(\frac{I_1'}{I_0} - \frac{I_1^2}{I_0^2}\right)\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta A_i}{\delta h_k} - \frac{A_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

$$C_{ik}^Y = \frac{\delta m_i^y}{\delta h_k} = \frac{B_i}{R_i}[q - |m|^2]\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta B_i}{\delta h_k} - \frac{B_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

Correlation functions :

$$
\begin{aligned}
C^X_{ik} &= f_1^{(i)}(q,m)\left[\sum_j J^R_{ij} C^x_{jk} - \sum_j J^I_{ij} C^y_{jk} + \delta_{ik}\right] + g^{(i)}(q,m)\left[\sum_j J^R_{ij} C^y_{jk} + \sum_j J^I_{ij} C^x_{jk}\right] \\
\tilde{C}^x_{ik} &= f_1^{(i)}(q,m)\left[\sum_j J^R_{ij} \tilde{C}^x_{jk} - \sum_j J^I_{ij} \tilde{C}^y_{jk}\right] + g^{(i)}(q,m)\left[\sum_j J^R_{ij} \tilde{C}^Y_{jk} + \sum_j J^I_{ij} \tilde{C}^x_{jk} + \delta_{ik}\right] \\
C^y_{ik} &= g^{(i)}(q,m)\left[\sum_j J^R_{ij} C^x_{jk} - \sum_j J^I_{ij} C^y_{jk} + \delta_{ik}\right] + f_2^{(i)}(q,m)\left[\sum_j J^R_{ij} C^y_{jk} + \sum_j J^I_{ij} C^x_{jk}\right] \\
\tilde{C}^y_{ik} &= g^{(i)}(q,m)\left[\sum_j J^R_{ij} \tilde{C}^x_{jk} - \sum_j J^I_{ij} \tilde{C}^y_{jk}\right] + f_2^{(i)}(q,m)\left[\sum_j J^R_{ij} \tilde{C}^y_{jk} + \sum_j J^I_{ij} \tilde{C}^x_{jk} + \delta_{ik}\right]
\end{aligned}
$$

$$
\begin{aligned}
f_1^{(i)}(q_i,m_i) &= \frac{q_i - |m_i|^2 + \mu_i^2 - q_i\mu_i^2}{1+\mu_i^2} \\
f_2^{(i)}(q_i,m_i) &= \frac{1 + \mu_i + \mu_i^2 - q_i - \mu_i|m_i|^2 - q_i\mu_i^2}{1+\mu_i^2} \\
g^{(i)}(q_i,m_i) &= \frac{2\sqrt{\mu_i}q_i - \sqrt{\mu_i}|m_i|^2 - \sqrt{\mu_i}}{1+\mu_i^2}
\end{aligned}
$$

$$C^x_{ik} = f_1^{(i)}\Gamma^A_{ik} + g^{(i)}\Gamma^B_{ik}$$
$$C^y_{ik} = g^{(i)}\Gamma^A_{ik} + f_2^{(i)}\Gamma^B_{ik}$$

$$\Gamma^A_{ik} = k_1^{(i)}C^x_{ik} - k_2^{(i)}C^y_{ik}$$
$$\Gamma^B_{ik} = k_3^{(i)}C^x_{ik} - k_4^{(i)}C^y_{ik}$$

$$k_1^{(i)} = \frac{f_2}{f_1 f_2 - g^2}, \quad k_2^{(i)} = \frac{g}{f_1 f_2 - g^2}, \quad k_3^{(i)} = \frac{g}{g^2 - f_1 f_2}, \quad k_4^{(i)} = \frac{f_1}{g^2 - f_1 f_2}$$

Next, solving :

$$\sum_j J^R_{ij}C^x_{jk} - \sum_j J^I_{ij}C^y_{jk} + \delta_{ik} = \Gamma^A_{ik}$$
$$\sum_j J^R_{ij}C^y_{jk} + \sum_j J^I_{ij}C^x_{jk} = \Gamma^B_{ik}$$

Interaction coupling :

$$J^R_{il} = \sum_k [(\Gamma^A_{ik} - \delta_{ik})(C^y_{kl})^{-1} + \Gamma^B_{ik}(C^x_{kl})^{-1}][\sum_k (C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})]^{-1}$$
$$J^I_{il} = \sum_k [-(\Gamma^A_{ik} - \delta_{ik})(C^x_{kl})^{-1} + \Gamma^B_{ik}(C^y_{kl})^{-1}][\sum_k (C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})]^{-1}$$

What's next?

- Numerical test
- We need data :
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--> estimate J

- XY model
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- Generic XY model

-- > Set up Monte Carlo simulations.

Generic XY model

Hamiltonian :

$$\mathcal{H} = -\sum_{\langle ij \rangle}[J^R_{ij}\cos(\phi_i-\phi_j) - J^I_{ij}\sin(\phi_i-\phi_j)] - \sum_i [h^R_i\cos(\phi_i) + h^I_i\sin(\phi_i)]$$

Applying the same approach :

$$\rho(\phi_i) = \frac{\exp(R_i\cos(\phi_i-\alpha_i))}{I_0(R_i)}$$

$$\begin{aligned} A_i &= \sum_j J^R_{ij}\langle\cos(\phi_j)\rangle - \sum_j J^I_{ij}\langle\sin(\phi_j)\rangle + h^R_i \\ B_i &= \sum_j J^R_{ij}\langle\sin(\phi_j)\rangle + \sum_j J^I_{ij}\langle\cos(\phi_j)\rangle + h^I_i \end{aligned}$$

$$R_i = \sqrt{(A_i^2+B_i^2)} \quad , \quad \alpha_i = \arctan\frac{B_i}{A_i}$$

Correlation functions

$$C^X_{ik} = \frac{\delta m^x_i}{\delta h^R_k} \quad , \quad \tilde{C}^X_{ik} = \frac{\delta m^x_i}{\delta h^I_k} \quad , \quad C^Y_{ik} = \frac{\delta m^y_i}{\delta h^R_k}, \quad \tilde{C}^Y_{ik} = \frac{\delta m^y_i}{\delta h^I_k} .$$

$$q_i = \langle\cos^2(\phi_i)\rangle \quad , \quad \frac{I_1'}{I_0} - \frac{I_1^2}{I_0^2} = q_i - |m_i|^2 \quad , \quad |m_i|^2 = (m^x_i)^2 + (m^y_i)^2 \quad , \quad \mu_i = \frac{m^y_i}{m^x_i}$$

$$\begin{aligned} \frac{A_i}{R_i} &= \cos\alpha_i = \cos[\arctan(\mu_i)] = \sqrt{\frac{1}{1+\mu_i^2}} \\ \frac{B_i}{R_i} &= \sin\alpha_i = \sin[\arctan(\mu_i)] = \sqrt{\frac{\mu_i}{1+\mu_i^2}} \end{aligned}$$

Magnetizations :

$$m^x_i = \frac{I_1(R_i)}{I_0(R_i)}\frac{A_i}{R_i} \quad ; \quad m^y_i = \frac{I_1(R_i)}{I_0(R_i)}\frac{B_i}{R_i}$$

Correlation functions :

$$C^X_{ik} = \frac{\delta m^x_i}{\delta h_k} = \frac{A_i}{R_i}\left(\frac{I_1'}{I_0} - \frac{I_1^2}{I_0^2}\right)\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta A_i}{\delta h_k} - \frac{A_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

$$C^Y_{ik} = \frac{\delta m^y_i}{\delta h_k} = \frac{B_i}{R_i}[q - |m|^2]\frac{\delta R_i}{\delta h_k} + \frac{I_1}{I_0 R_i}\frac{\delta B_i}{\delta h_k} - \frac{B_i}{R_i^2}\frac{\delta R_i}{\delta h_k}$$

Correlation functions :

$$\begin{aligned} C^X_{ik} &= f_1^{(i)}(q,m)\left[\sum_j J^R_{ij}C^x_{jk} - \sum_j J^I_{ij}C^y_{jk} - \delta_{ik}\right] + g^{(i)}(q,m)\left[\sum_j J^R_{ij}C^y_{jk} - \sum_j J^I_{ij}C^x_{jk}\right] \\ \tilde{C}^X_{ik} &= f_1^{(i)}(q,m)\left[\sum_j J^R_{ij}\tilde{C}^x_{jk} - \sum_j J^I_{ij}\tilde{C}^y_{jk}\right] + g^{(i)}(q,m)\left[\sum_j J^R_{ij}\tilde{C}^y_{jk} + \sum_j J^I_{ij}\tilde{C}^x_{jk} - \delta_{ik}\right] \\ C^Y_{ik} &= g^{(i)}(q,m)\left[\sum_j J^R_{ij}C^x_{jk} - \sum_j J^I_{ij}C^y_{jk} + \delta_{ik}\right] + f_2^{(i)}(q,m)\left[\sum_j J^R_{ij}C^y_{jk} - \sum_j J^I_{ij}C^x_{jk}\right] \\ \tilde{C}^Y_{ik} &= g^{(i)}(q,m)\left[\sum_j J^R_{ij}\tilde{C}^x_{jk} - \sum_j J^I_{ij}\tilde{C}^y_{jk}\right] + f_2^{(i)}(q,m)\left[\sum_j J^R_{ij}\tilde{C}^y_{jk} + \sum_j J^I_{ij}\tilde{C}^x_{jk} - \delta_{ik}\right] \end{aligned}$$

$$\begin{aligned} f_1^{(i)}(q_i,m_i) &= \frac{q_i - |m_i|^2 + \mu_i^2 - q_i\mu_i^2}{1+\mu_i^2} \\ f_2^{(i)}(q_i,m_i) &= \frac{1 + \mu_i + \mu_i^2 - q_i - \mu_i|m_i|^2 - q_i\mu_i^2}{1+\mu_i^2} \\ g^{(i)}(q_i,m_i) &= \frac{2\sqrt{\mu_i}q_i - \sqrt{\mu_i}|m_i|^2 - \sqrt{\mu_i}}{1+\mu_i^2} \end{aligned}$$

What's next?

- Numerical test
- We need data :
- Magnetizations, correlation functions
  --> estimate J

- XY model
- Critical behavior of the XY model in complex topologies , Miguel Ibáñez Berganza, Luca Leuzzi, Phys. Rev. B 88, 144104 (2013)

- Generic XY model

  -- > Set up Monte Carlo simulations.

$$\begin{aligned} C^x_{ik} &= f_1^{(i)}\Gamma^A_{ik} + g^{(i)}\Gamma^B_{ik} \\ C^y_{ik} &= g^{(i)}\Gamma^A_{ik} + f_2^{(i)}\Gamma^B_{ik} \end{aligned} \quad \rightarrow \quad \begin{aligned} \Gamma^A_{ik} &= k_1^{(i)}C^x_{ik} - k_2^{(i)}C^y_{ik} \\ \Gamma^B_{ik} &= k_3^{(i)}C^x_{ik} - k_4^{(i)}C^y_{ik} \end{aligned}$$

$$k_1^{(i)} = \frac{f_2}{f_1 f_2 - g^2}, \quad k_2^{(i)} = \frac{g}{f_1 f_2 - g^2}, \quad k_3^{(i)} = \frac{g}{g^2 - f_1 f_2}, \quad k_4^{(i)} = \frac{f_1}{g^2 - f_1 f_2}$$

Next, solving :

$$\begin{aligned} \sum_j J^R_{ij}C^x_{jk} - \sum_j J^I_{ij}C^y_{jk} + \delta_{ik} &= \Gamma^A_{ik} \\ \sum_j J^R_{ij}C^y_{jk} + \sum_j J^I_{ij}C^x_{jk} &= \Gamma^B_{ik} \end{aligned}$$

Interaction coupling :

$$J^R_{il} = \sum_k[(\Gamma^A_{ik} - \delta_{ik})(C^y_{kl})^{-1} + \Gamma^B_{ik}(C^x_{kl})^{-1}][\sum_k(C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})]^{-1}$$

$$J^I_{il} = \sum_k[-(\Gamma^A_{ik} - \delta_{ik})(C^x_{kl})^{-1} + \Gamma^B_{ik}(C^y_{kl})^{-1}][\sum_k(C^x_{ik}(C^y_{kl})^{-1} + C^y_{ik}(C^x_{kl})^{-1})]^{-1}$$

In Collaboration with Dr. Andrea Pagnani(HuGeF Torino)

MC simulation

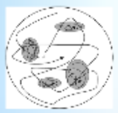
Technical Details :

MC simulation + Parallel Tempering (Exchange MC)

- MC simulation run in parallel for desired range of temperatures.
- NPT copies of the system with same realization
- Exchanges are done according to :

$$A = \begin{cases} e^{-(\beta_{low}-\beta_{high})\Delta E} & \text{if } \Delta E > 0 \\ 1 & \text{otherwise.} \end{cases}$$

- if swap increases E_low --> unlikely
- otherwise --> most likely
- high temperature :ergodic and thus helps low temp system to sample from whole phase space



In Metropolis algorithm :

- For each site a random angle is chosen and updates are done according to update rules.

Continuous variables :
--> CUDA programming on GPU
--> effective implementation

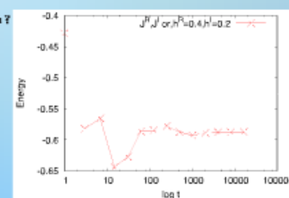
GPU : nVidia GeForce GTX 680

What do we extract ?

Equilibrium measures of thermodynamic properties : correlations and magnetisation(of our interest)

How do we know we reached equilibrium ?

Check energy to be constant in log time scale.



Underlying graphs ?
2D, 3D lattice, Levy graphs,
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--> Set topology

P(J) ?
Ordered, Bimodal
Disordered(J : Gaussian) $P(J_{ij}) = \frac{1}{\sqrt{2\pi}}\exp(\frac{-J_{ij}^2}{2})$

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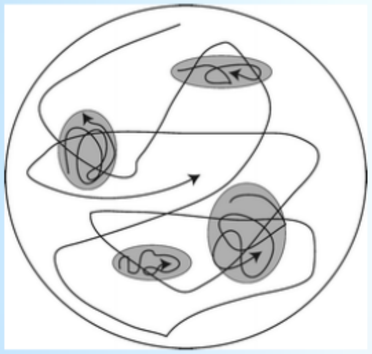

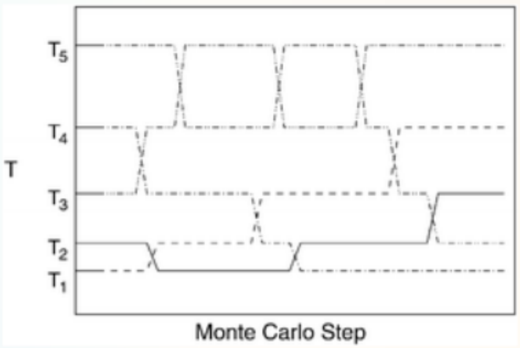

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__device__ float **__fsqrt_rn** (float x)
Compute $\sqrt{x}$ in round-to-nearest-even mode.

__device__ float **__fsqrt_ru** (float x)
Compute $\sqrt{x}$ in round-up mode.

__device__ float **__fsqrt_rz** (float x)
Compute $\sqrt{x}$ in round-towards-zero mode.

__device__ float **__log10f** (float x)
Calculate the fast approximate base 10 logarithm of the input argument.

__device__ float **__log2f** (float x)
Calculate the fast approximate base 2 logarithm of the input argument.

__device__ float **__logf** (float x)
Calculate the fast approximate base e logarithm of the input argument.

__device__ float **__powf** (float x, float y)
Calculate the fast approximate of x^y.

__device__ float **__saturatef** (float x)
Clamp the input argument to [+0.0, 1.0].

__device__ void **__sincosf** (float x, float *sptr, float *cptr)
Calculate the fast approximate of sine and cosine of the first input argument.

__device__ float **__sinf** (float x)
Calculate the fast approximate sine of the input argument.

__device__ float **__tanf** (float x)

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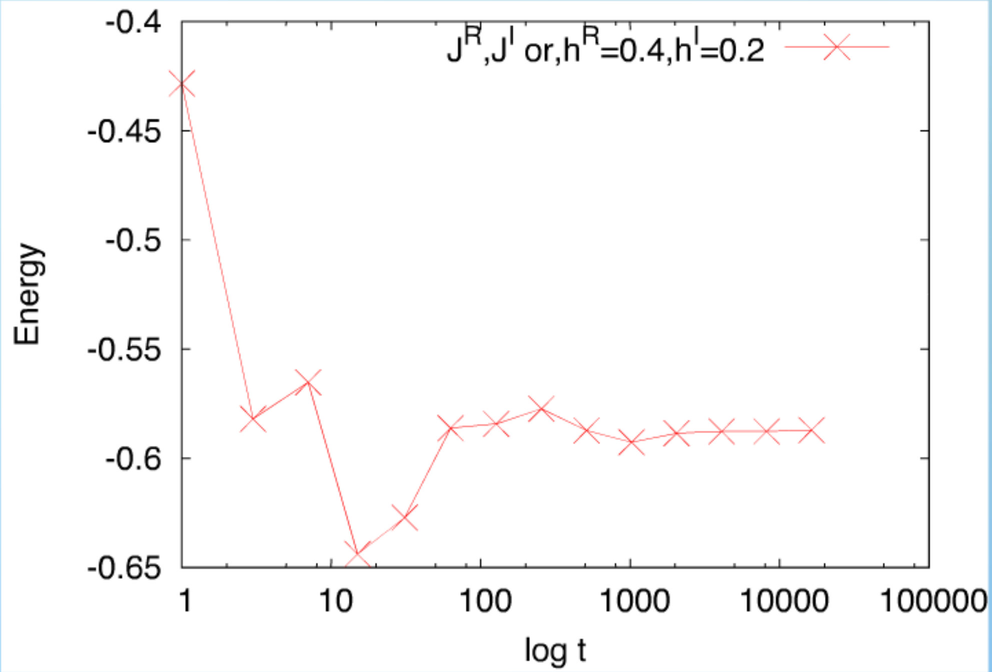

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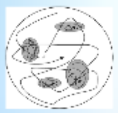

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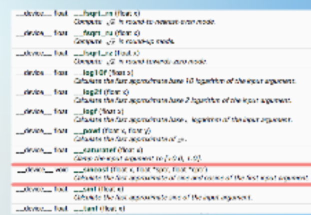

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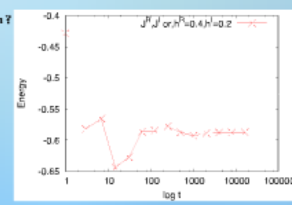

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Data Analysis

System under scrutiny :

$$\mathcal{H} = -\sum_{\langle i,j\rangle} [J^R_{ij}\cos(\phi_i - \phi_j) - J^I_{ij}\sin(\phi_i - \phi_j)] - \sum_i [h^R_i \cos(\phi_i) + h^I_i \sin(\phi_i)]$$

Aim :

In search of critical temperature and critical exponents.

Techniques :

- **Monte Carlo simulation**
- **Parallel Tempering**

- MC simulations done for underlying topology as Erodos Renyi graphs with average connectivity c = 6.
- For various lattice size with length N = 64, 256, 576 and 1024.
- Temperature range : 1.5 - 10

Further work in MC simulation

- Data from this MC simulation is used for testing inference techniques further.
- Jacknife method for error estimation for uncorrelated data.
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FORMULAS

Ensemble average and topological average : $\langle O\rangle = \frac{1}{N_{ER}}\frac{1}{t_{MCS}}\sum_{k=1}^{N_{ER}}\sum_{t=1}^{t_{MCS}} O_{k,t}$

Specific heat $c = \frac{1}{N}\frac{\partial\langle H\rangle}{\partial T} = \frac{1}{NT^2}(\langle H^2\rangle - \langle H\rangle^2)$

Susceptibility $\chi = N(\langle m^2\rangle - \langle m\rangle^2)$

Binder Cumulant $U_4 = \frac{\langle(m - \langle m\rangle)^4\rangle}{\langle(m - \langle m\rangle)^2\rangle^2} - 1$

Magnetization

$$m_x = \frac{1}{N}\sum_{i=1}^{N}\cos(\phi_i)$$
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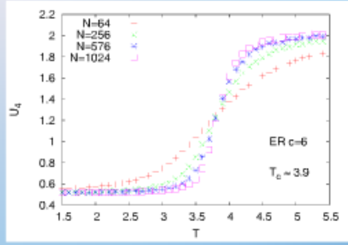
RESULT

Binder cumulant for J ordered:

System Jr-Ji and h=0



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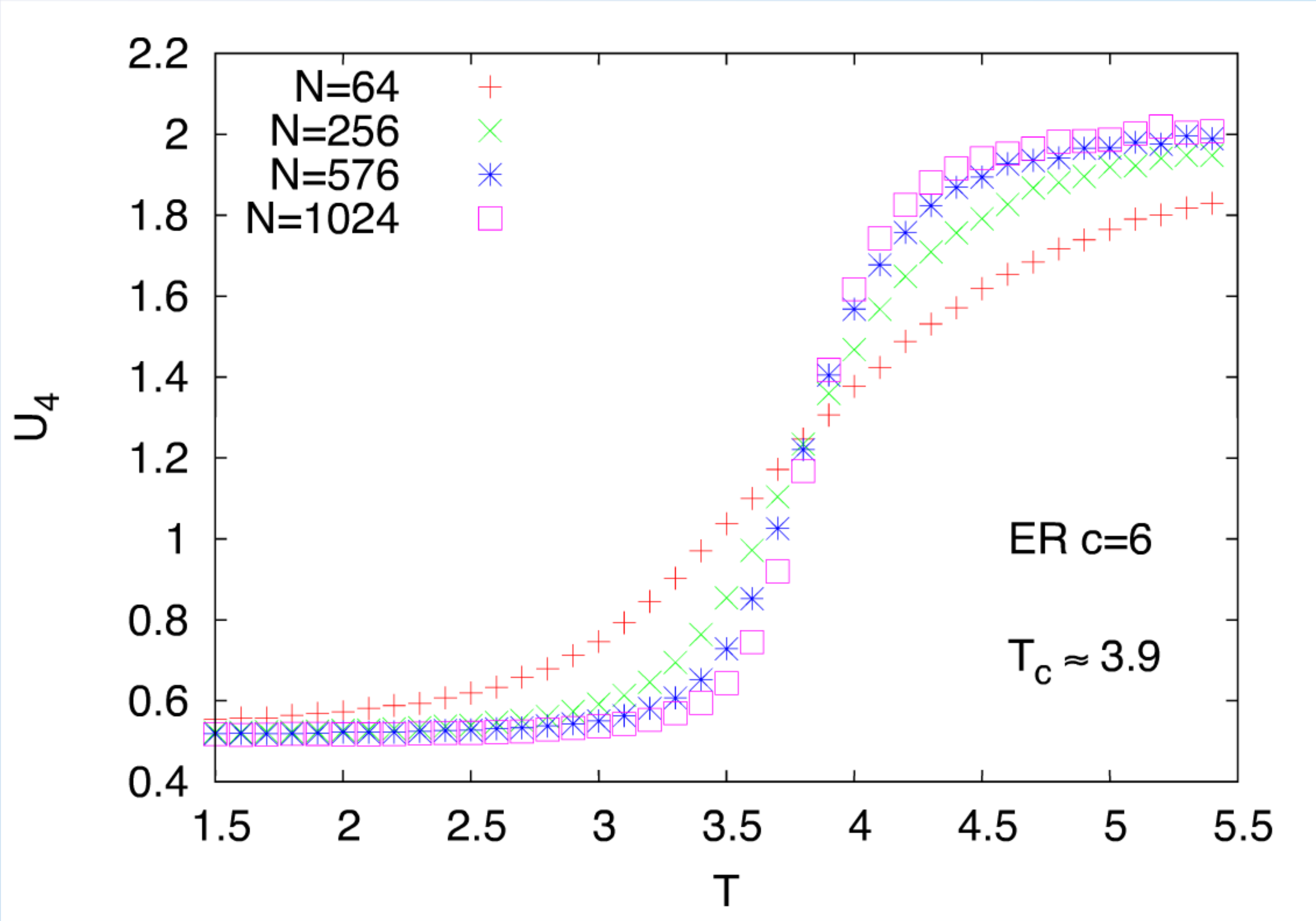

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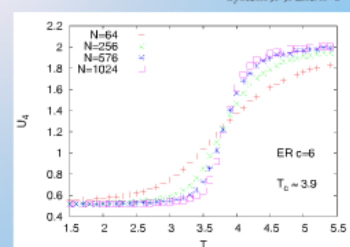
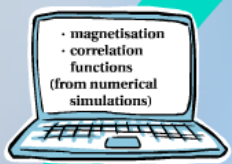


RESULTS

Numerical Test



Interaction coupling J

Starting with the real distribution of J's: ordered, disordered

magnetization and correlations

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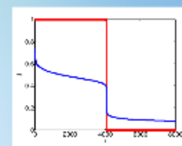
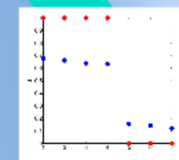
Infer J and compare with Jtru

Reconstruction of interaction couplings : XY model

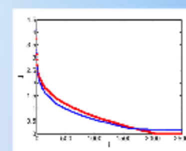
Erdös Rènyi c = 6, N = 1024, T = 1.4 (Tc = 1.43)

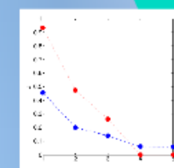
Ordered :





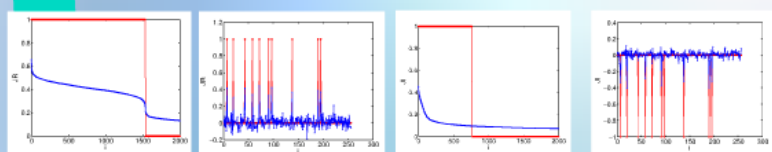
Disordered :
Tc = 0.983





Ordered :

JR - JI with hr = 0.4 and hi = 0.2



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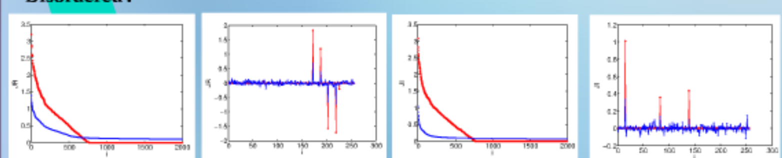
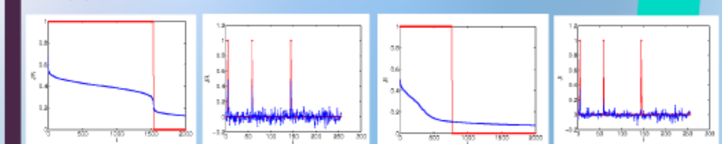


Generic XY model

Ordered JR-JI for c=6 N=256 with hr = hi =0



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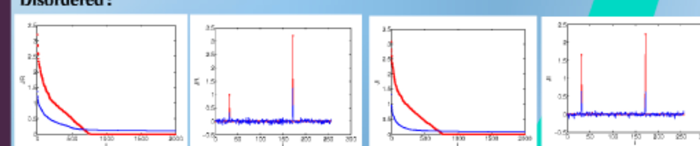

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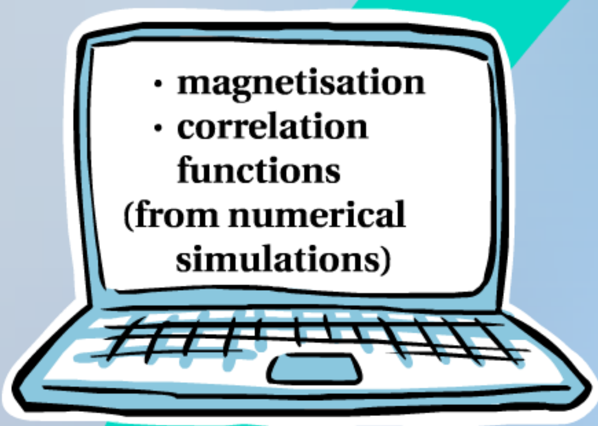

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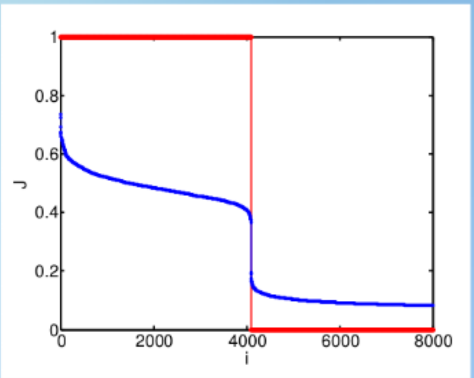

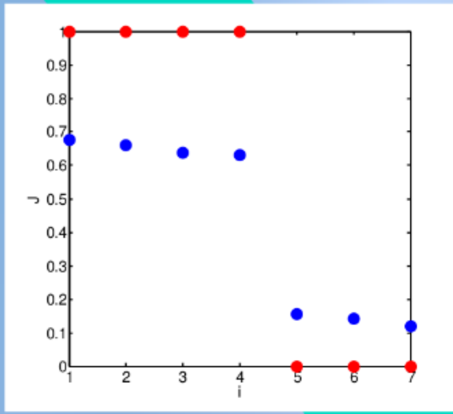

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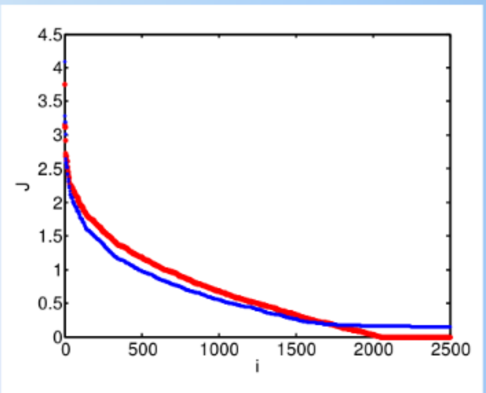

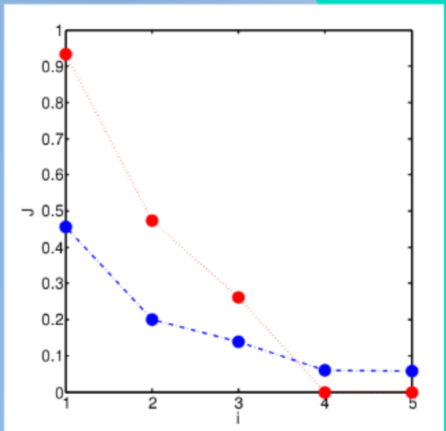

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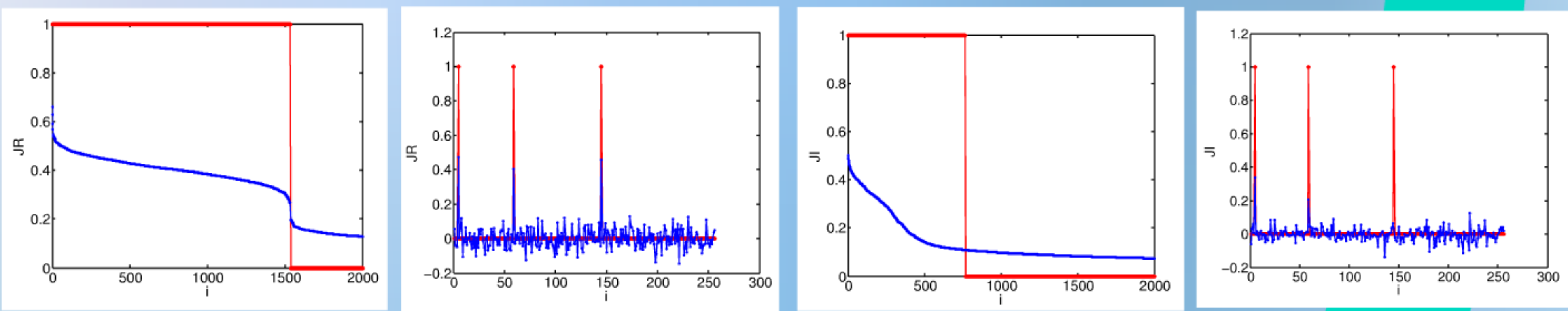

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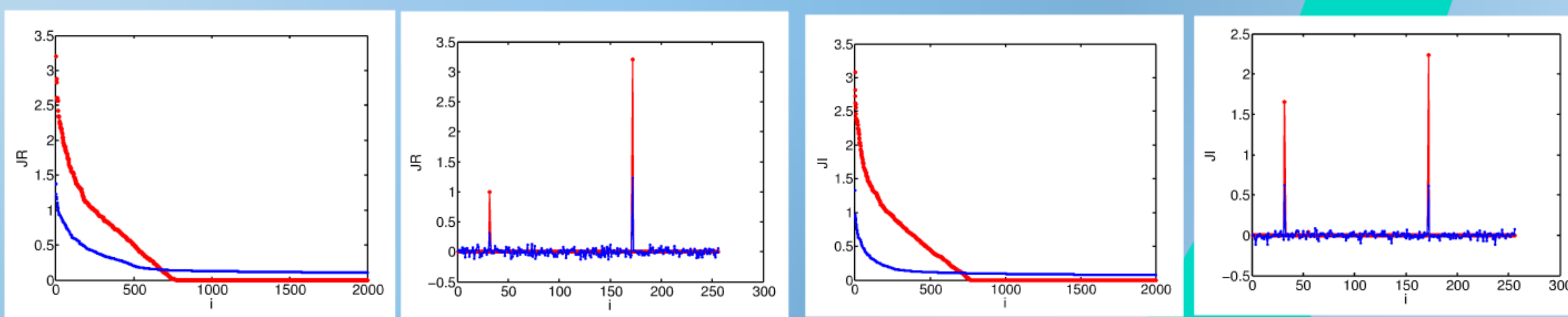

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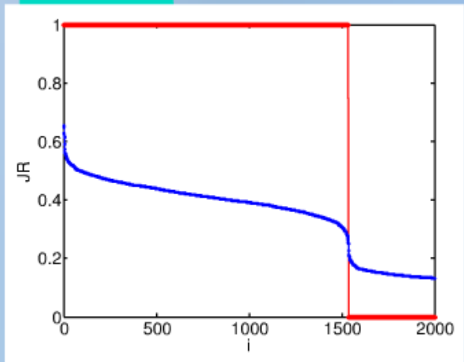

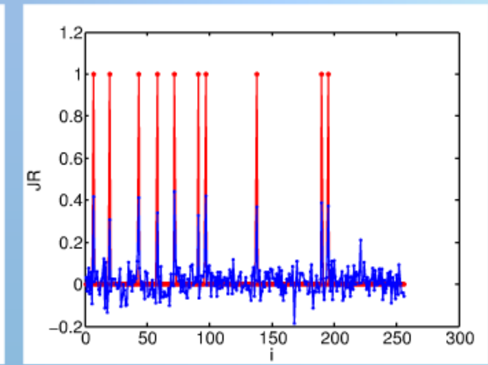

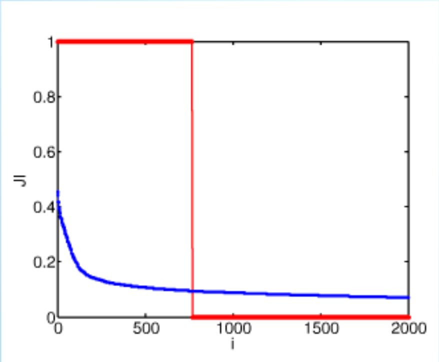

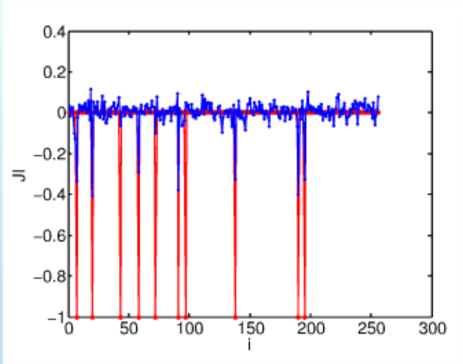

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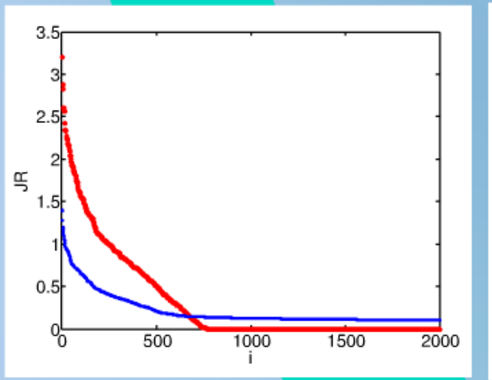
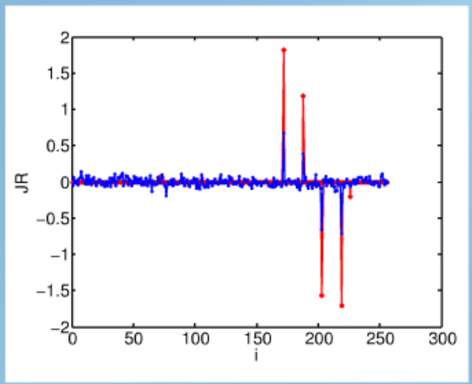
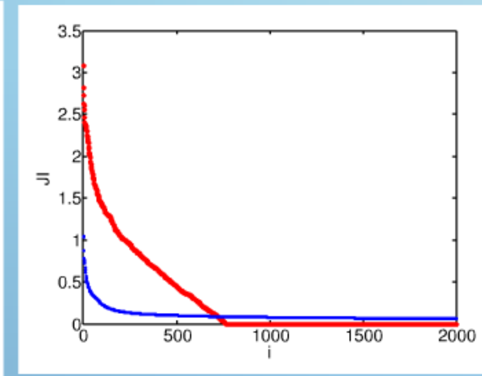
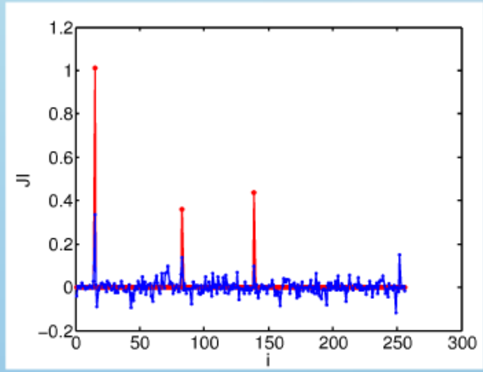


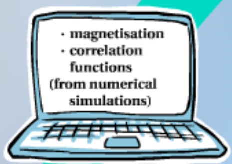






RESULTS

Numerical Test



Interaction coupling J

Starting with the real distribution of J's: ordered, disordered

magnetization and correlations

$$C^X_{ij} = \langle \cos(\phi_i - \phi_j) \rangle \qquad m_i^x = \langle \cos(\phi_i) \rangle$$
$$C^Y_{ij} = \langle \sin(\phi_i - \phi_j) \rangle \qquad m_i^y = \langle \sin(\phi_i) \rangle$$

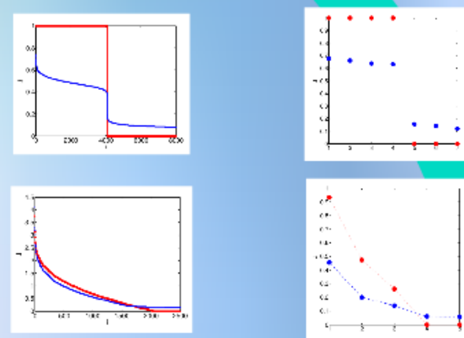
Infer J and compare with Jtru

Reconstruction of interaction couplings : XY model

Erdös Rènyi c = 6, N = 1024, T = 1.4 (Tc = 1.43)

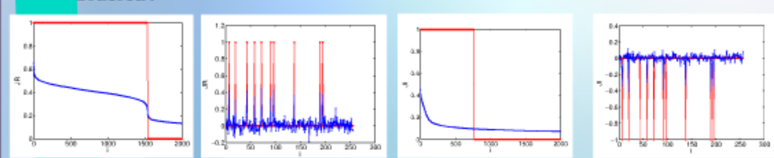
Ordered :

Disordered :
Tc = 0.983

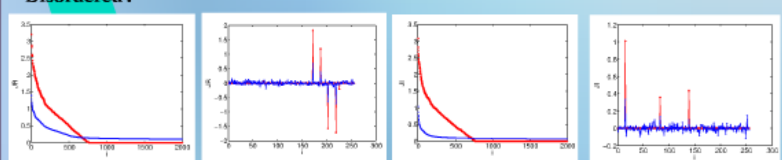


Ordered :

JR - JI with hr = 0.4 and hi = 0.2

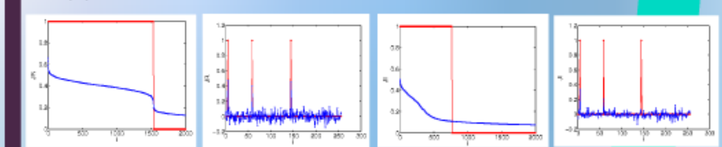


Disordered :

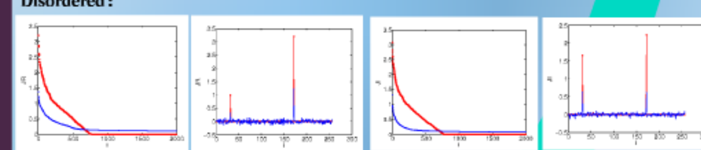


Generic XY model

Ordered JR-JI for c=6 N=256 with hr = hi =0



Disordered :



Prezi

Further work in Inference

In continuation...

1) Qualitative and quantitative analysis by :

- Sensitivity Plots
- Receiving Operating Characteristic(ROC)

2) Pseudo Likelihood Method:

- local alternative to maximum likelihood estimation for networks

Next, data from experiments:

Interaction coupling J

- We can infer the J's but we won't have the initial distribution to compare with.

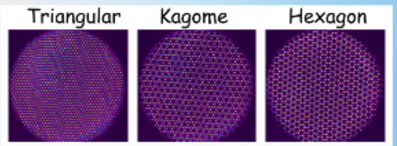
In collaboration with experimental group of Prof. Claudio Conti, ISC-CNR

--> to 4-XY model where apart from 2 point correlation function, we need to consider also the 4 -body correlation function.

$$C^{(4)}_{jklm} = \langle \cos(\phi_j - \phi_k + \phi_l - \phi_m) \rangle$$

--> Apply to spherical spin model, where we can look at the intensities and therefore spectral behavior. We also investigate Intensity correlation function on top of phase correlation function.

Thousands of coupled lasers



$$H = -\sum_{i<j} J_{ij} \cos(\phi_i - \phi_j)$$

- Instead of mode coupling, there is coupling between different independent lasers.
- Mapping phase of each laser --> angular orientation of planar spins.
- J's are ordered since they are all ordered cavity lasers. We can quantitatively estimate the interaction coefficient.

Micha Nixon, Eitan Ronen, Asher A. Friesem and Nir Davidson, *Observing geometric frustration with thousands of coupled lasers*, PRL 110, 184102 (2013)

Cavity Method

XY generic model

Techniques :
- Cavity Method on bipartite graph
- Belief Propagation
- Population dynamics
- optimized programming using CUDA on GPU's

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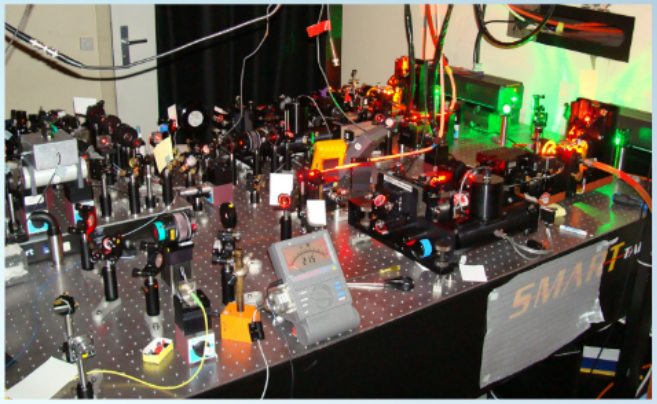

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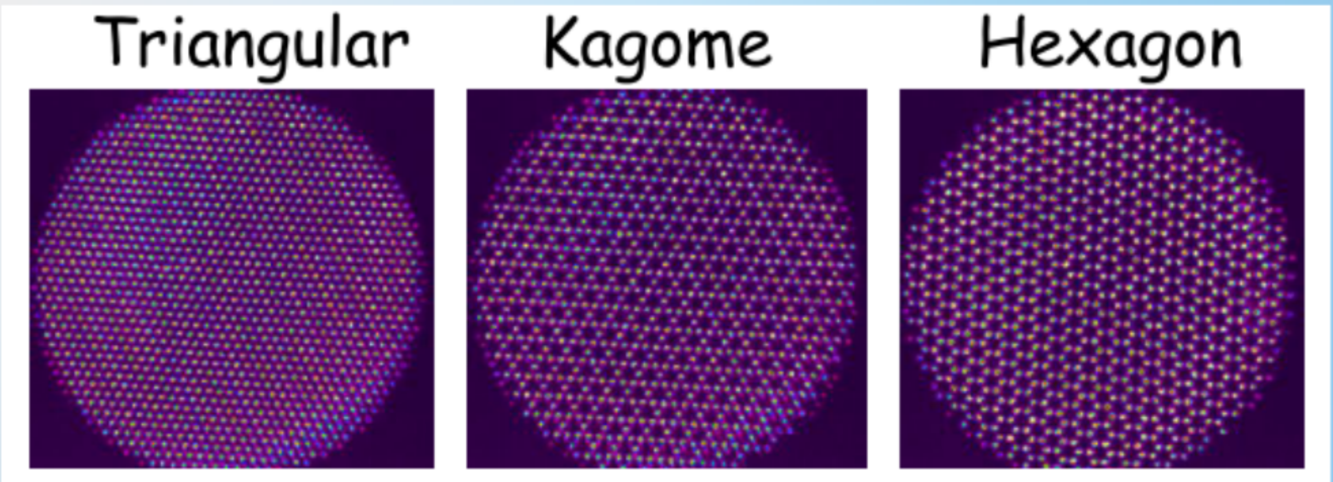

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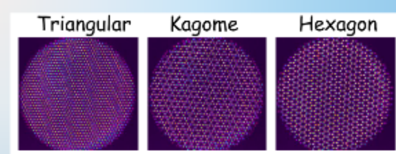

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Thank you for your attention.

"Inference of coupling of waves in non linear disordered medium."

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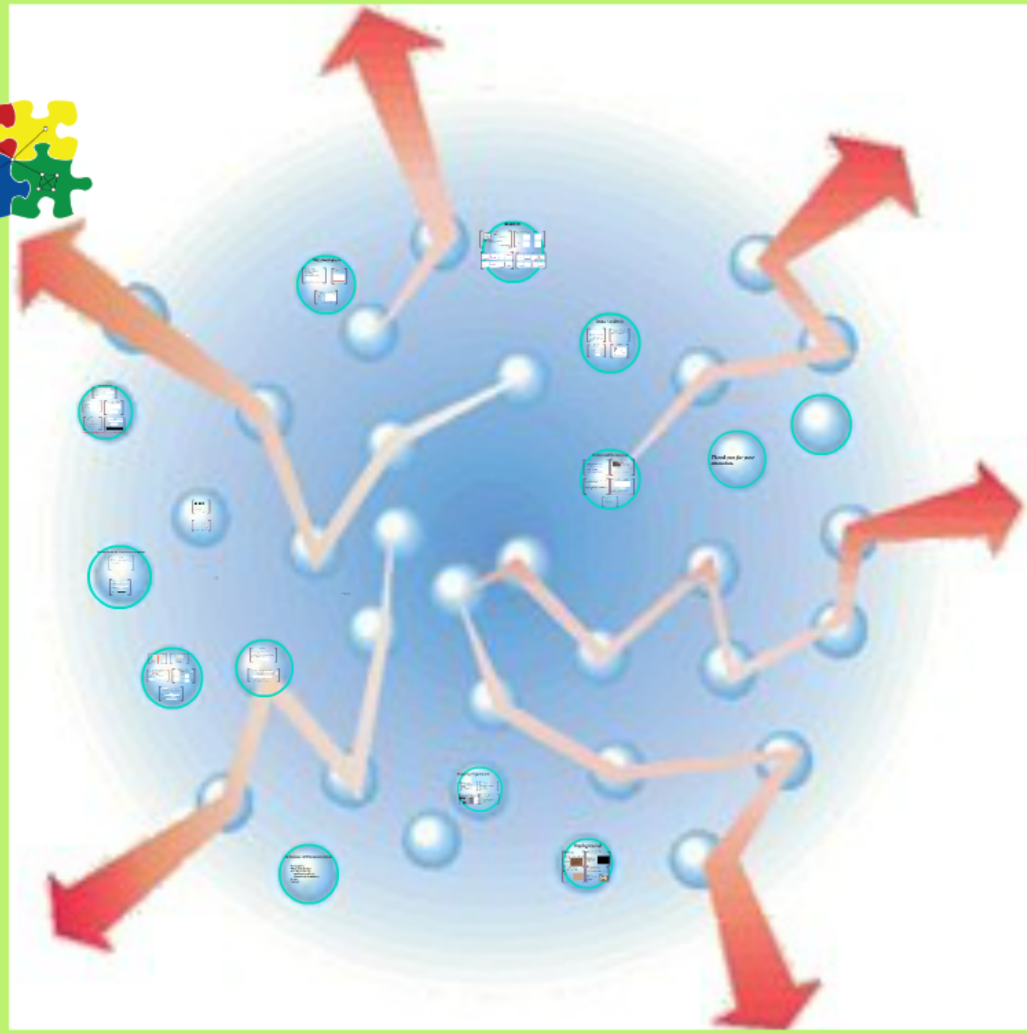
MARIE CURIE ACTIONS

SEVENTH FRAMEWORK PROGRAMME



IPCF

Consiglio Nazionale delle Ricerche

SAPIENZA UNIVERSITÀ DI ROMA

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