Contagion in an interacting economy

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Introduction

What are we trying to do?

Understanding economic crises and economic stability via disordered systems methods

In the beginning

- Hatchett & Kühn 2006 : *Effect of economic interactions on credit risk* presents a simple model of economic interaction and study the contagion effects.
- The contagion mechanism is cast as a linear threshold model with noise
- Some mean-field results, but few results for sparse or heterogeneous networks

Model setting

- We take a weighted graph of size N
- Each node has an initial wealth θ_i, and a state $n_{i,t} = 0, 1$ ("active" and "defaulted")
- Each edge has weights $(w_{ij}, w_{ji}) \sim p_w(w_{ij}, w_{ji})$
- Every time a neighbor j defaults, node i loses w_{ij} from his wealth.

Model setting

- at time t, a node defaults $(n_{t-1} = 0, n_t = 1)$ with some transition probability W_t taken to be of the form
$$W_t \left(\frac{1}{\sigma_\xi} \left[\sum_{j \in \partial j} n_{j,t-1} w_{ij} - \theta_i - \xi_{0,t} \right] \right)$$
- $\xi_{0,t}$: system-wide bias $\rightarrow$ "global economic condition"
- Reference case: $W_t(x) = \Phi(x), \quad \xi_{0,t} = \xi_0, \quad \sigma_\xi = 1$
- we write
$$\mathbf{n}(t) = (0, \cdots, 0, \underset{\substack{\uparrow \\ t}}{1}, 1, \cdots)$$

Simple model: large connectivity

- Large-but-dilute connectivity limit with Gaussian interactions and noise
- Erdös-Renyi case : narrow degree distribution, largely equivalent to regular networks
  $\rightarrow$ only the wealth appears as a node disorder
- wealth and connectivity patterns are uncorrelated $\rightarrow$ uniform (large) sampling at each node

$\rightarrow$ Dynamics are equivalent to that of the complete graph (Corollary: all unbiased connectivity systems whose minimum degree is large enough are largely equivalent)

Effective dynamics on the complete graph

default rate at given wealth

$$n_{t+1}(\theta) = n_t(\theta) + [1 - n_t(\theta)]\,\Phi\left(\frac{\overline{w}\,m_t - \theta - \xi_0}{\sqrt{\sigma_\xi^2 + \sigma_w^2 m_t}}\right) \qquad (1)$$

averaged default rate

$$m_{t+1} = m_t + \left\langle [1 - n_t(\theta)]\,\Phi\left(\frac{\overline{w}\,m_t - \xi_0 - \theta}{\sqrt{\sigma_\xi^2 + \sigma_w^2 m_t}}\right)\right\rangle_\theta \qquad (2)$$

Cavity solution

What happens with small degrees?

- we want to compute m_t for (sparse) random graphs
- we consider a node i with degree k_i and initial wealth θ_i
- What is the marginal default probability at time t of this node ?

Cavity solution : key insights

Consider a central node defaulting at t_i:

- before t_i, it doesn't influence its neighbors
- after t_i, its neighbors do not influence it

$\rightarrow$ no memory effects

Cavity solution : reasoning I

- if the graph is a tree, we can write

$$p_i(t_i) = \sum_{\{\tau_j\}_{j\in\partial i}} p_i(t_i|\,\{\tau_j\}_{j\in\partial i}) \prod_{j\in\partial i} p_j(\tau_j|t_i)$$

- and likewise

$$p_j(\tau_j|t_i) = \sum_{\{\tau_l\}_{l\in\partial j\setminus i}} p_j(\tau_j|t_i,\{\tau_l\}_{l\in\partial j\setminus i}) \prod_{l\in\partial j\setminus i} p_l(\tau_l|\tau_j)$$

- if a neighbor defaults after the node, the default time has no importance

$$\forall\tau' > \tau, \qquad p(\tau|\tau') = p(\tau|\tau) \equiv \rho(\tau)$$

Cavity solution : reasoning II

- Similarly,

$$\forall \tau > t, \qquad p(t|\tau, \tau_2, \cdots, \tau_n) = p(t|t, \tau_2, \cdots, \tau_n)$$

- hence $\forall r \in \partial j \backslash i$,

$$\sum_{\tau_r} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\backslash i}) \prod_{l\in\partial j\backslash i} p_l(\tau_l|\tau_j) =$$

$$\sum_{\tau_r<\tau_j} p_j(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\backslash i})\rho_r(\tau_r) \prod_{l\in\partial j\backslash\{i,r\}} p_l(\tau_l)$$

$$+ p_j\left(\tau_j|t_i, \{\tau_l\}_{l\in\partial j\backslash i}\right)\left(1 - \sum_{\tau_r<\tau_j} \rho_r(\tau_r)\right) \prod_{l\in\partial j\backslash\{i,r\}} p_l(\tau_l|\tau_j)$$

Cavity : conclusion

- Main point : everything can be computed from the ρ, which follow a forward-integration relation

$$\rho(\tau) = \sum_k \frac{kp(k)}{\langle k \rangle} \sum_{\tau_1,\cdots,\tau_{k-1}\in\{1,\cdots,\tau\}} \prod_{l|\tau_l<\tau} \rho(\tau_l) \prod_{l|\tau_l=\tau} \left[1 - \sum_{\tau'<\tau} \rho(\tau_l)\right]$$
$$\times \left\langle P\left(\tau|\theta, \sum_{l=1}^{k-1} w_l \mathbf{n}(\tau_l)\right)\right\rangle_{\theta,\mathbf{w}} ,$$

where

$$P(t|\theta, \mathbf{h}) = W_{t-1}\left(h_{t-1} - \theta\right) \prod_{s=0}^{t-2} \left[1 - W_s\left(\theta - h_s\right)\right]$$

Cavity : result

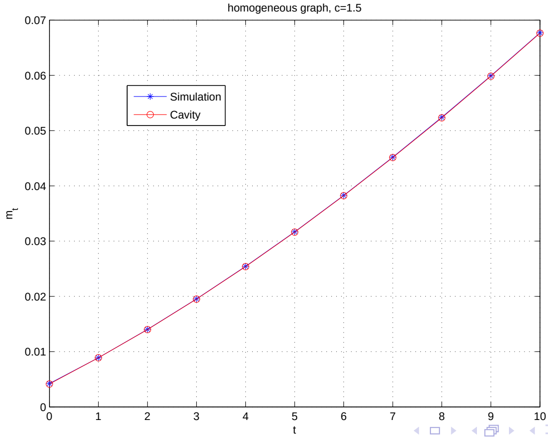
- and from there,

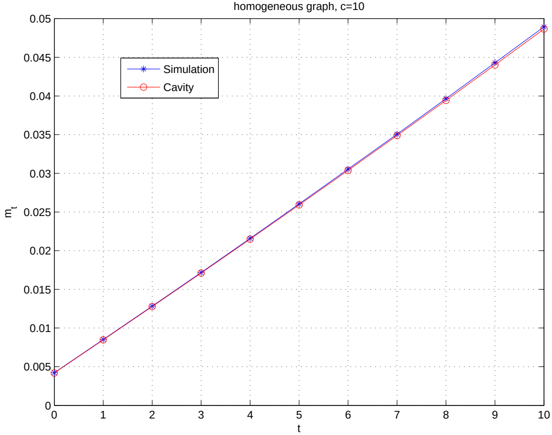
$$p(t) = \sum_k p(k) \sum_{\substack{\tau_1,\cdots,\tau_k \\ \in\{1,\cdots,t\}}} \prod_{l|\tau_l<t} \rho(\tau_l) \prod_{l|\tau_l=t} \left[1 - \sum_{\tau'<t} \rho(\tau_l)\right]$$
$$\times \left\langle P\left(t|\theta, \sum_{l=1}^{k-1} w_l \mathbf{n}(\tau_l)\right)\right\rangle_{\theta,\mathbf{w}},$$

Conclusion

Forward integration instead of fixed-point equation
$\rightarrow$ Easily done numerically

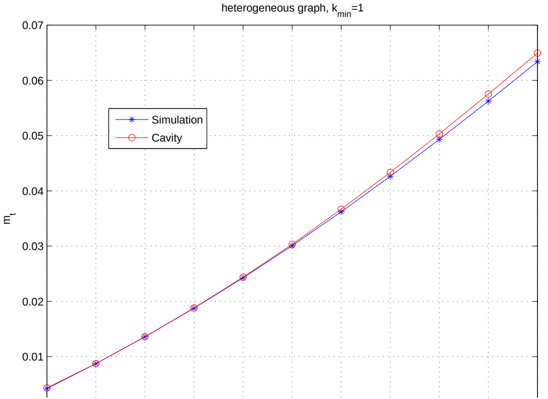
numerical results : homogeneous networks



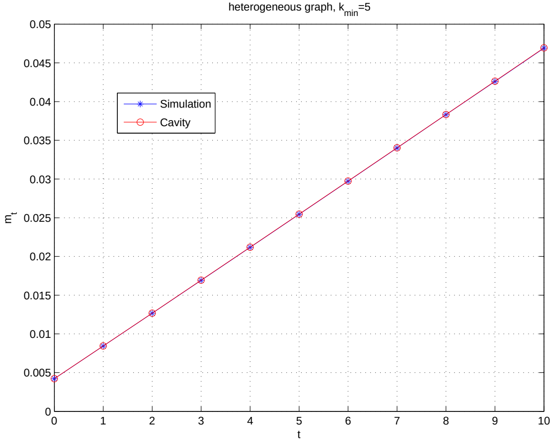


mean defaulted fraction as a function of time for a homogeneous network, $c = 10$

numerical results : heterogeneous networks

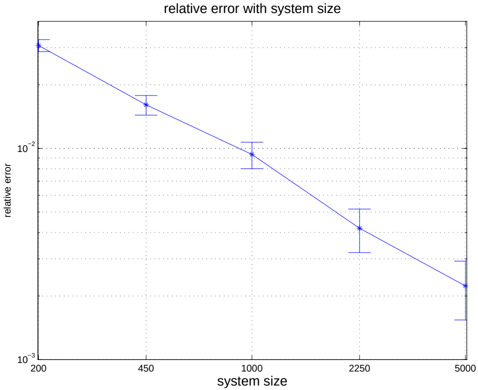


mean defaulted fraction as a function of time for a heterogeneous network, $\langle k \rangle = 1.3$



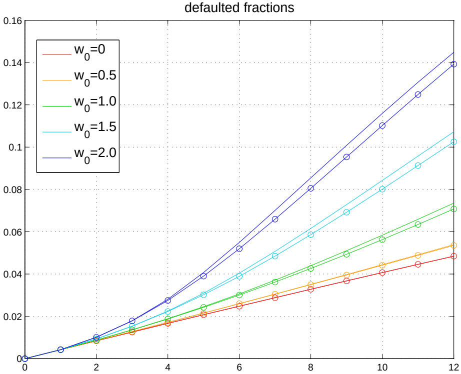
mean defaulted fraction as a function of time for a heterogeneous network, $\langle k \rangle = 10$

numerical results : network size



deviation from cavity for different network sizes

numerical results : interaction strength



Defaulted fractions for different mean interaction strength : simulation (circled) and theory. Network size is set at $N = 500$.

extensions and new directions

From this point, many different fields of investigations

current research directions

- Study of rare events and large deviations
- Interaction with other contagion channels → asset overlap contagion
- Inclusion of recovery aspects → extension to SIR model
- Examination of regulatory policy: do CCPs (Central Counter-Party clearing house) limit systemic risk?

introduction

Two related questions

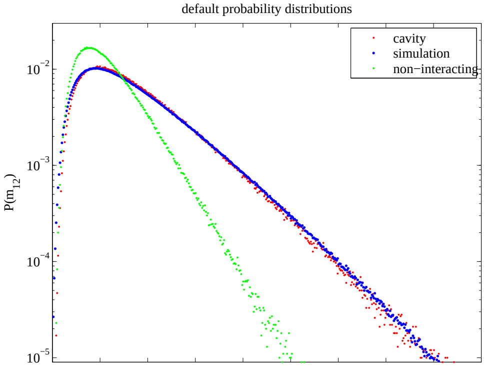
- what is the probability of a large-scale crisis?
- How sensitive is the model to external shocks/bias?

Effect of macro-economic bias

How sensitive is the economy to a downturn?

- Many ways to bias the trajectories toward default or survival
- in our model: ξ_0 parameter represents global economic conditions$\rightarrow$ $\xi_0 \sim \mathcal{N}(0, 0.2)$
- distribution over ξ_0 induces a distribution over the defaulted fraction
- right tail of the distribution $\rightarrow$ sensitivity of the economy to a (large) downturn

macro-economic forcing



End-of-year defaulted fraction distribution (induced): simulation (blue), cavity (red), non-interacting case (green)

Crisis distribution

What is the intrinsic probability of a large-scale crisis ?

- Numerically: difficult to probe the tail of the default distribution
- Analytically: compute the rate function via the Gartner-Ellis theorem

→ compute $\mu(\psi) = \frac{1}{N} \log \left\langle \exp \psi \sum_i n_{i,T} \right\rangle$

→ another way of biasing the trajectories toward default

- problem: solving the equations is not obvious

simple case

- Simplest case: identical wealth, regular graph, fixed couplings, two time steps
- can derive low-ψ, high-ψ expansion easily
- for the complete rate function: Newton method
- scaling : $\sim T^2$, $\times$ sampling needed for desired precision for random processes (e.g. random couplings), $\times$ number of steps needed for convergence (depends on wanted precision)

$\rightarrow$ limiting factor: if high precision is needed, can only accommodate limited sources of disorder

analytic equations

$$\begin{cases} x = & \dfrac{rx^{k-1} + (1-r)y^{k-1} + (1-r)f_x(y,z)\left(e^{\psi}-1\right)}{rx^{k} + (1-r)y^{k} + (1-r)f(y,z)\left(e^{\psi}-1\right)} \\[2ex] y = & \dfrac{rx^{k-1} + (1-r)y^{k-1} + (1-r)f_y(y,z)\left(e^{\psi}-1\right)}{rx^{k} + (1-r)y^{k} + (1-r)f_0(y,z)\left(e^{\psi}-1\right)} \\[2ex] z = & \dfrac{rx^{k-1}}{rx^{k} + (1-r)y^{k} + (1-r)f_0(y,z)\left(e^{\psi}-1\right)} \end{cases} \tag{3}$$

Gartner-Ellis theorem

$$\mu(\psi) \equiv \frac{1}{N} \log \left\langle \exp \psi \sum_i n_{i,T} \right\rangle \quad (4)$$

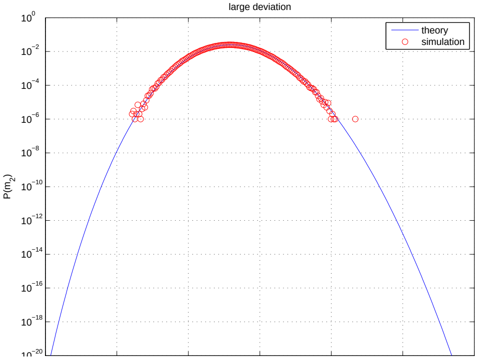
$$= \log \left[r x^k + (1-r) y^k + (1-r) f_0(y,x) \left(e^\psi - 1 \right) \right] \quad (5)$$

$$- \frac{k}{2} (y^2 + 2z(x-y) - 1)$$

Gartner-Ellis theorem

$$\log P_N(m) \simeq N \inf_\psi \{ \mu(\psi) - m\psi \}$$

result



default probability at $T = 2$: large deviation prediction (blue) and simulations (red) for $N = 500$

current questions

annealed computation for a non self-averaging problem

- simple case: not a problem (only one homogeneous regular tree, error needs more than one time step)
- if more complicated: lesser agreement, but is it still usable
- the heuristics of self-averaging-or-not are unclear

scaling

Can we make the computation scale well with more time steps and disorder ?

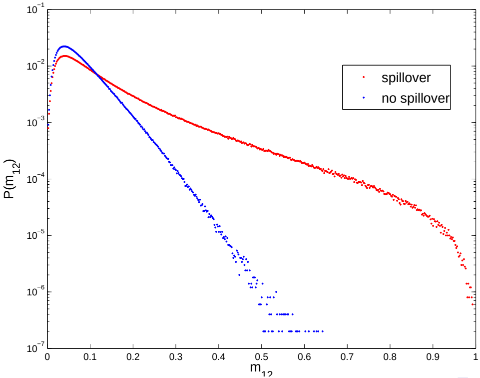
Spillover effects

- Interesting extension of the model: inclusion of asset overlap contagion
- A firm, short on liquidity, sells a large amount of assets in a short time → asset price drops → wealth position drops for everyone holding this asset: $\theta_i \to \theta_i' < \theta_i$.
- in our model : only one asset class, firm sells when $\theta_{i,t} < f_c \times \theta_i$
- $$\theta_{i,t} = r(d)\,\theta_i - \sum_i w_{ij} c_{ij} n_{j,t} - \xi_{0,t}$$

  with $r(\mathbf{d}) = (1 + r_0\, d_t)^{-1}$,
- d_t : fraction of distressed firms at time t.

fire-sales : numerical comparison

Fire-sales $\rightarrow$ dramatic contagion enhancement



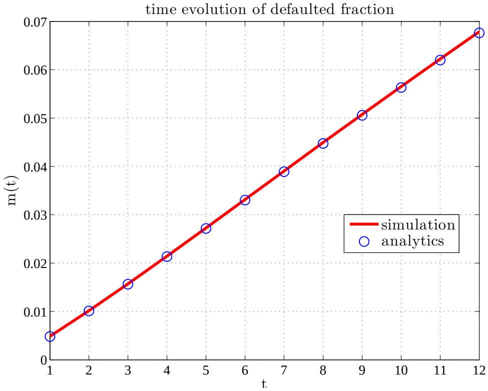
ongoing work

current questions

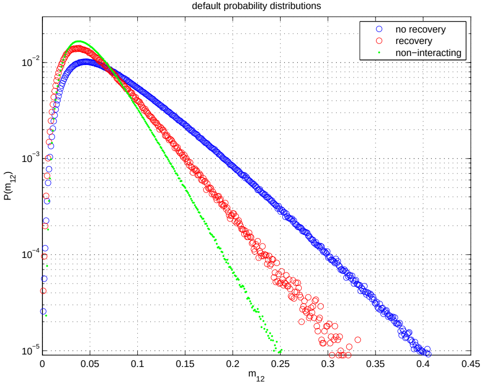
- What happens with more than one asset class?
- What happens with many more asset classes?
- How do we get better heuristics?

introduction

- It is possible to add any number of features to the model as long as they do not affect irreversibility or insensitivity to future states
- e.g. : recovery of lost funds with various set recovery scenario is possible
- restriction : scenario chosen cannot depend on future state (field post default), but can depend on field at default.
- example : neighbors recover $n\%$ of lost funds every time step after default, $n \sim \mathcal{U}[\![0, 50]\!]$



mean defaulted fraction: simulations (blue circles) and analytic predictions (red)



End-of-year default probabilities without recovery (blue), with recovery (red, $n_{max} = 50$), and without interaction(green)

Coming Soon (Sorry!)

Current issues

many unanswered questions

- It's unclear what the limitations of our annealed computations are
- lots of numerics, but few heuristics
- unrealistic networks, and difficult to improve on it

Thank you !