Ergodicity breaking in p-spin glass models and FPU problem

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Gino Del Ferraro

KTH Royal Institute of Technology, Stockholm

OUTLINE:

- **Brief overview on the FPU problem**
- **2+p-spin spherical models**

  Investigate metastable states:

    - Effective Potential Method
    - Disorder and Replicas
    - Looking for minima

  Dynamics:

    - evolution of correlation and response functions

- **3-spin spherical model partitioned in two interacting subsystems**

Brief overview on FPU problem

$$H = \sum_{i=0}^{N} \left[\frac{p_i^2}{2m} + \frac{K}{2} \big(q_{i+1} - q_i \big)^2 + \frac{\epsilon}{\alpha} \big(q_{i+1} - q_i \big)^{\alpha} \right]$$

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for $\epsilon = 0$ ⟶ **Integrable**

Using the normal modes

$$a_k = \sqrt{\frac{2}{N+1}} \sum_n q_n \sin\left(\frac{n\,k\,\pi}{N+1}\right) \qquad (k = 1, \ldots, N)\,,$$

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→ N non-interacting harmonic oscillators

frequencies

$$\omega_k = 2\sqrt{\frac{K}{m}}\,\sin\left(\frac{k\,\pi}{2(N+1)}\right)$$

energies

$$E_k = \frac{1}{2}\left[\left(\frac{\mathrm{d}a_k}{\mathrm{d}t}\right)^2 + \omega_k^2 a_k^2\right] = const\,.$$

equipartition law

$$\langle E_k \rangle = \frac{E_{tot}}{N}$$

Brief overview on FPU problem

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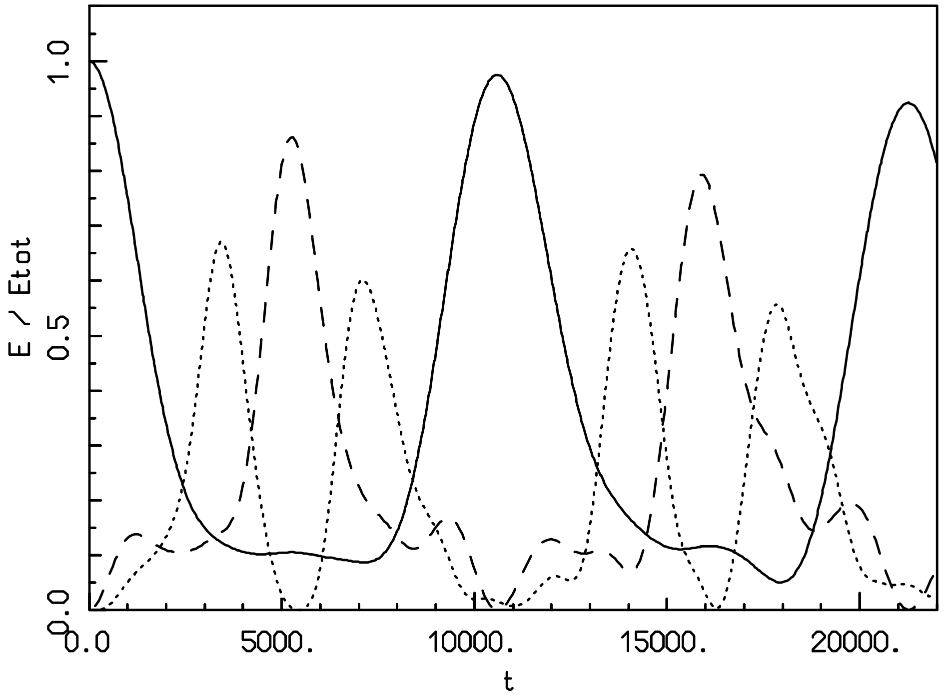

Fig. 14.1 Normalized modes energies $E_k(t)/E_{tot}$ for $k = 1$ (solid line), $k = 2$ (dashed line) and $k = 3$ (dotted line) obtained with $N = 32$, $\alpha = 3$ and $\epsilon = 0.1$. The initial condition is $E_1(0) = E_{tot} = 2.24$ and $E_k(0) = 0$ for $k = 2, \ldots, 32$. [Courtesy of G. Benettin]

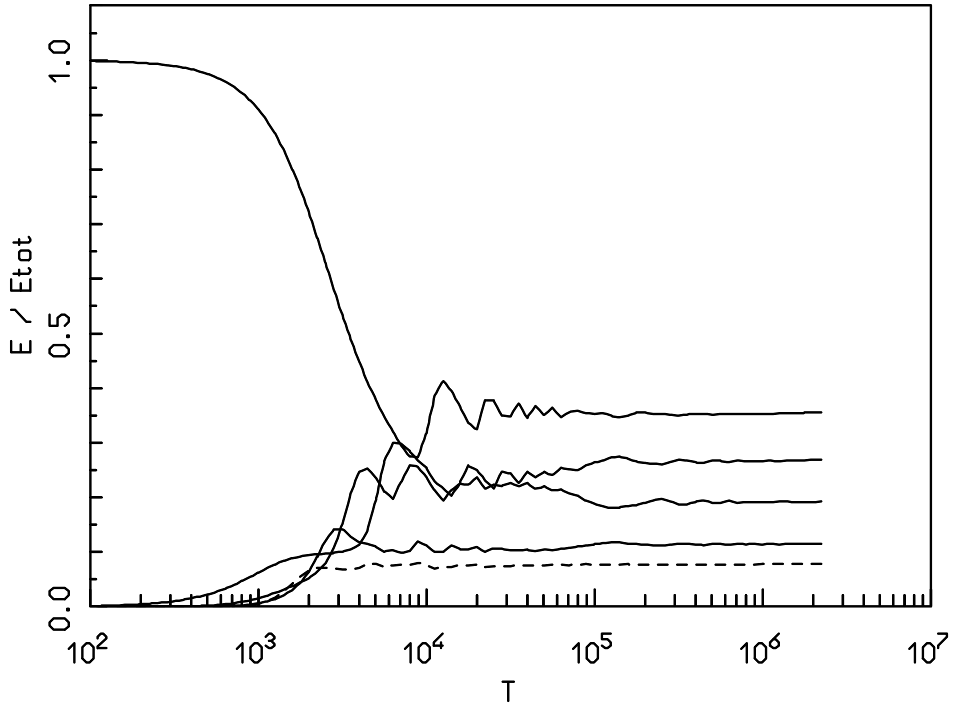

Fig. 14.2 Time averaged fraction of energy, in modes $k = 1, 2, 3, 4$ (bold lines, from top to below), the dashed line shows the time average of the sum from $k = 5$ to $N = 32$. The parameters of the system are the same as in Fig. 14.1. [Courtesy of G. Benettin]

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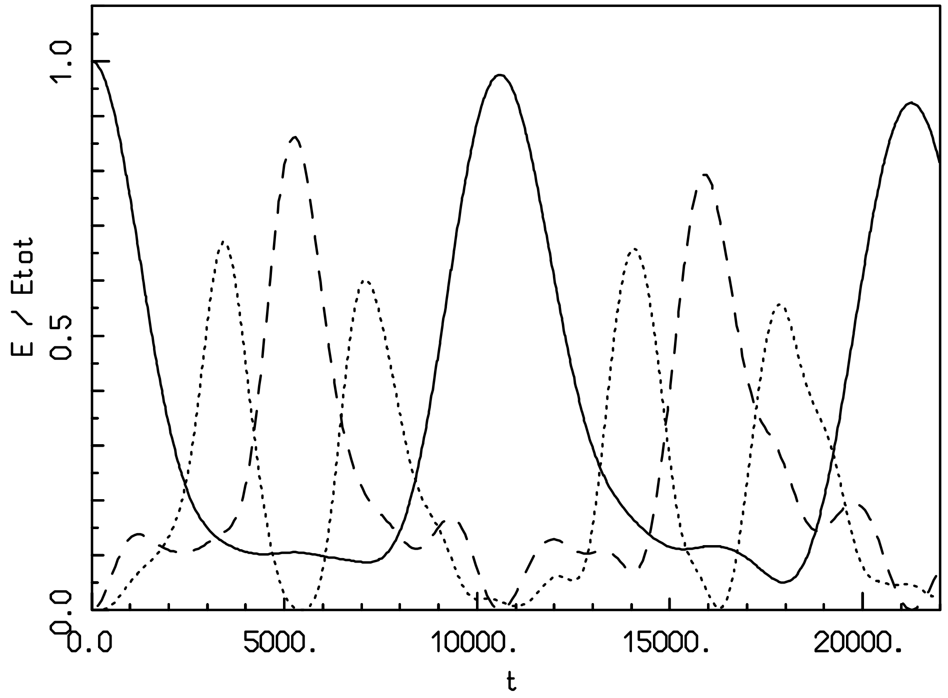

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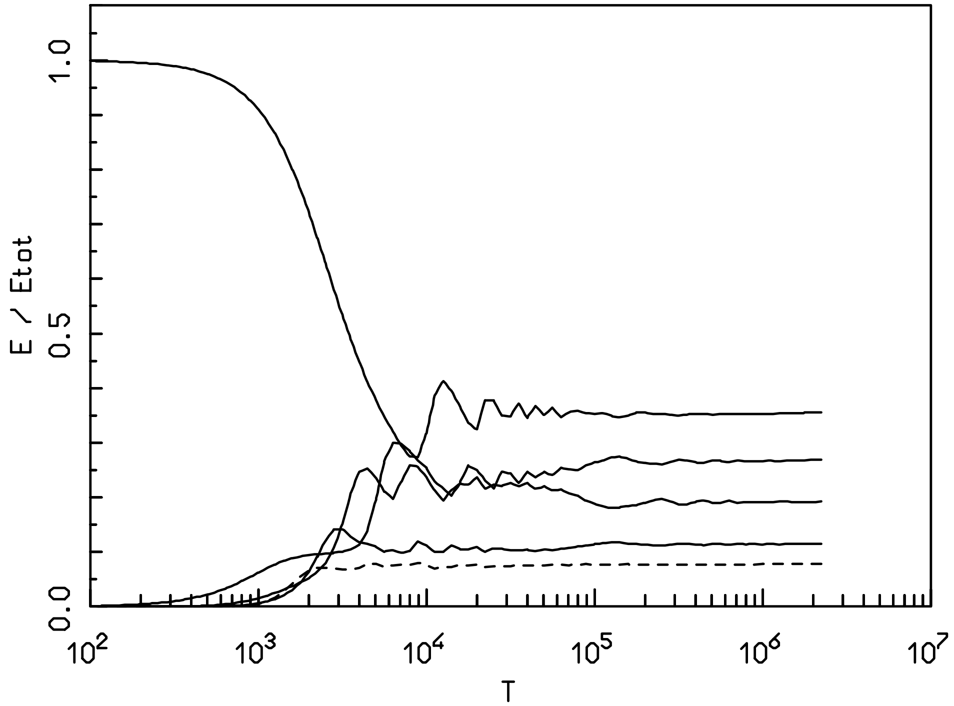

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Fermi-Pasta-Ulam problem

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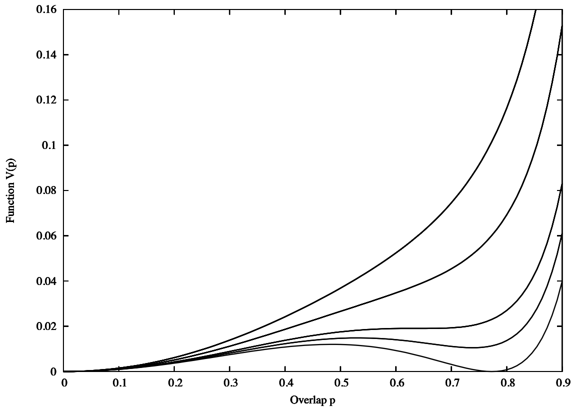
2+4 p-spherical spin Hamiltonian

$$H = \frac{1}{2} \sum_i p_i^2 - \sum_{i<j} J_{ij}\, s_i s_j - \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l + \frac{\mu_x(t)}{2} (\sum_i s_i^2 - N)$$

2+4 P-spin spherical model

- Effective Potential Method -



Potential Method

Two systems (the same) at two **different** temperatures

Reference system

$$P(\underline{s}) = \frac{\exp\left(-\beta' H_J[\underline{s}]\right)}{Z(\beta')}$$

Probe system

$$P(\underline{\sigma}) = \frac{\exp\left(-\beta H_J[\underline{\sigma}]\right)}{Z(\beta)}$$

Overlap: $Q(\underline{s}, \underline{\sigma}) = \dfrac{1}{N} \sum_i s_i \sigma_i$

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Constrained free-energy

$$F(\underline{s},\beta,\tilde{p}) = \lim_{N\to\infty} -\frac{1}{\beta N}\ln\int d\underline{\sigma}\,\mathrm{e}^{-\beta \mathrm{H}[\underline{\sigma}]}\,\delta(\tilde{\mathrm{p}} - \mathrm{Q}(\underline{\mathrm{s}},\underline{\sigma}))$$

Potential Function

$$V(\tilde{p},\beta,\beta') = \lim_{N\to\infty} -\frac{1}{\beta N}\overline{\int d\underline{s}\,\frac{\mathrm{e}^{-\beta' \mathrm{H}[\underline{\mathrm{s}}]}}{Z(\beta')}\ln\int d\underline{\sigma}\,\mathrm{e}^{-\beta \mathrm{H}[\underline{\sigma}]}\,\delta(\tilde{\mathrm{p}} - \mathrm{Q}(\underline{\mathrm{s}},\underline{\sigma}))}$$

[Franz, S., & Parisi, G. (1995). Recipes for metastable states in spin glasses. *Journal de Physique I*, *5*(11), 1401-1415]

Meaning of the minima

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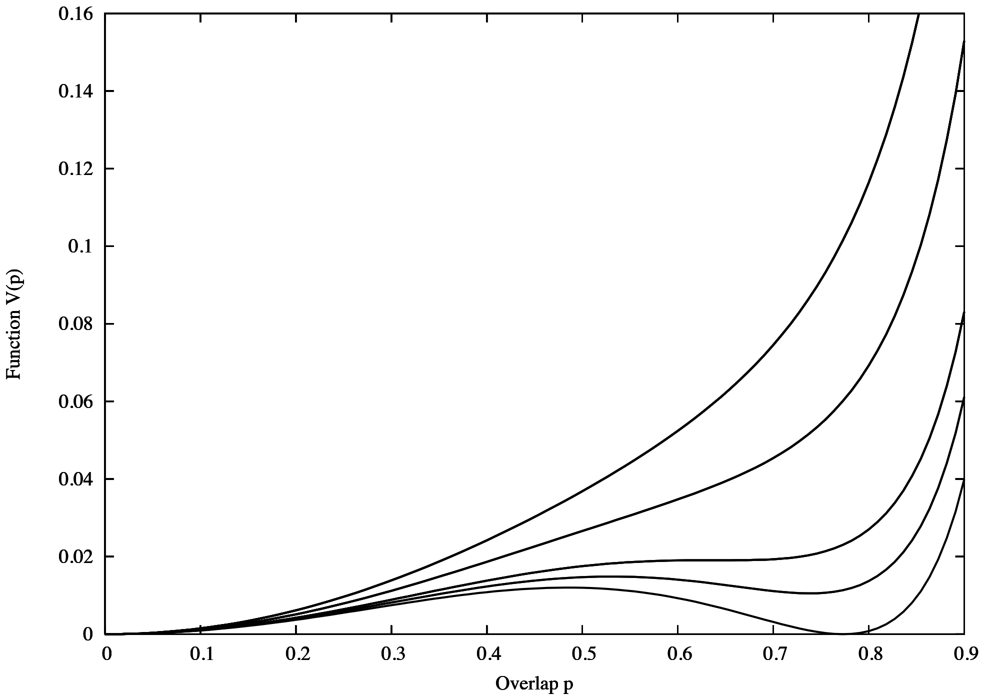

$$V(p,T)-F(T)$$

[S. Franz and G. Semerjian. *Analytical approaches to time and length scales in models of glasses* in *Dynamical heterogeneities in glasses, colloids and granular materials.* Oxford University Press, 2011.]

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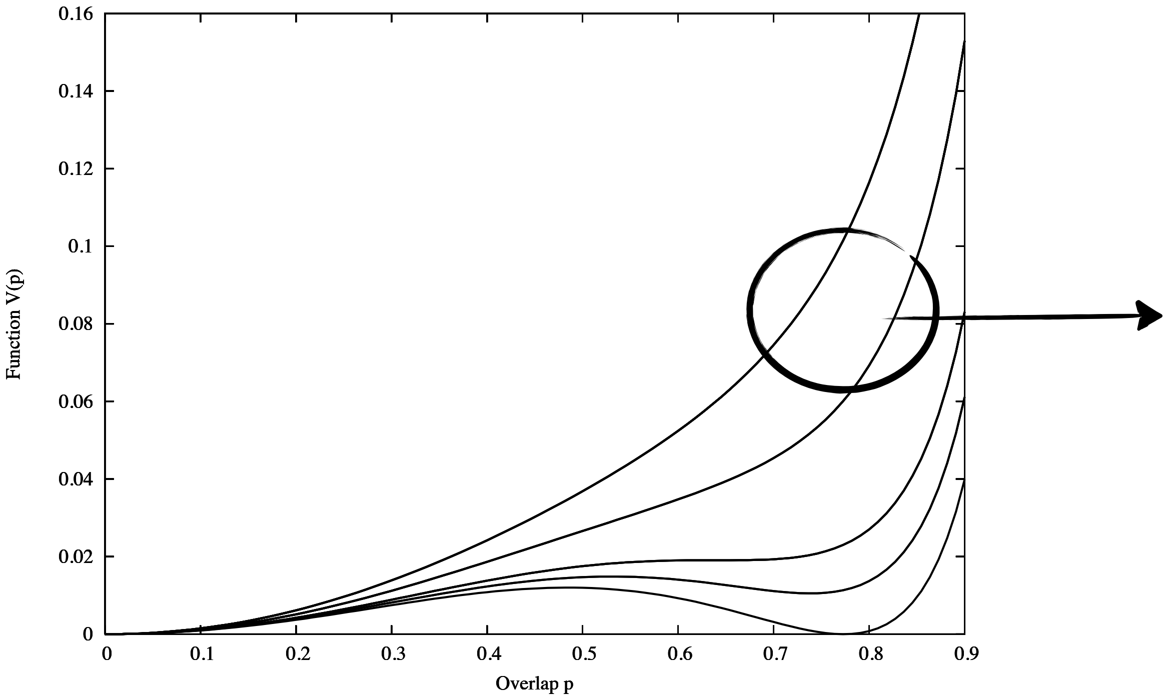

$V(p,T) - F(T)$

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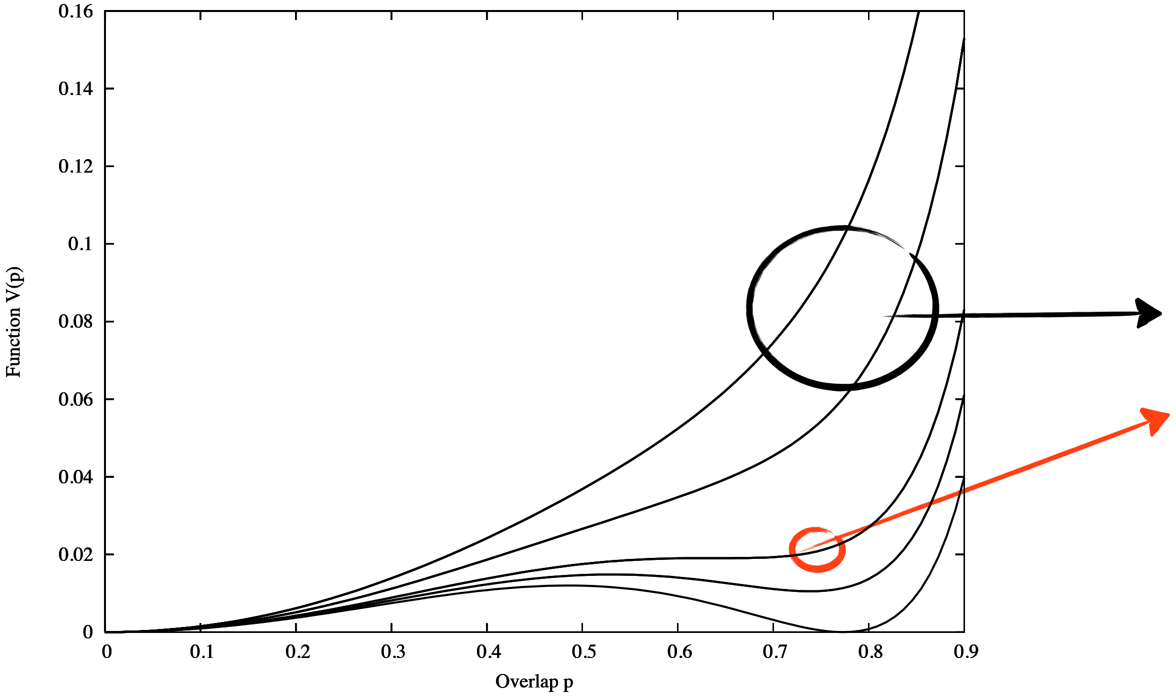

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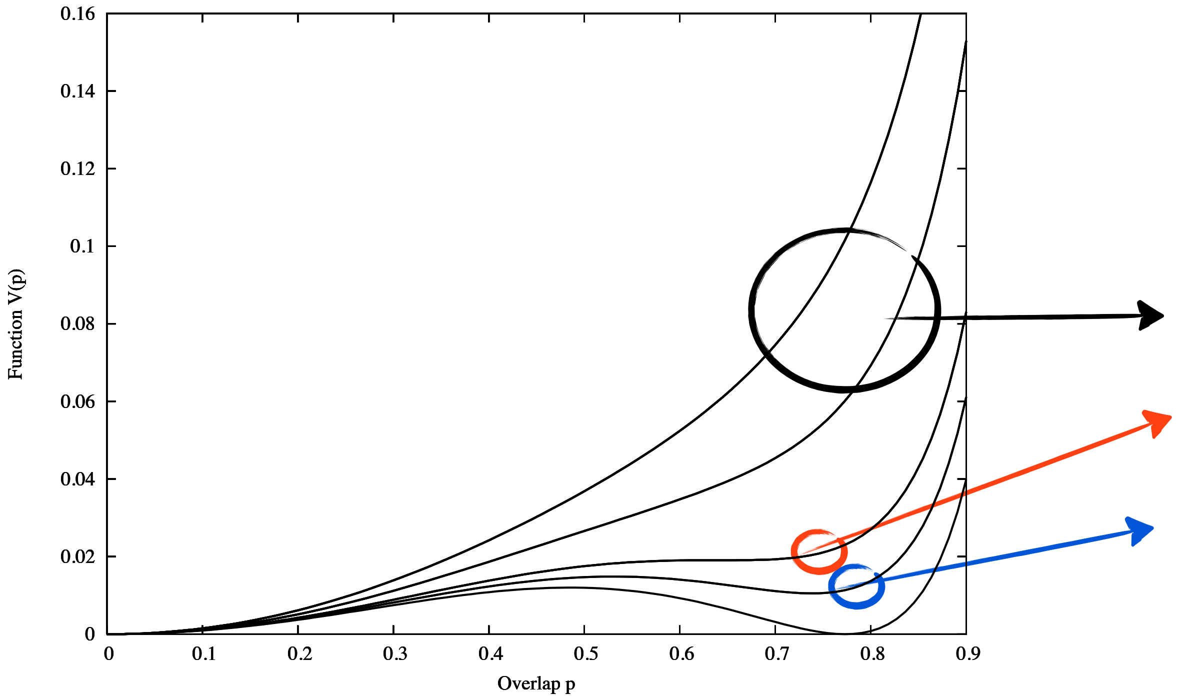

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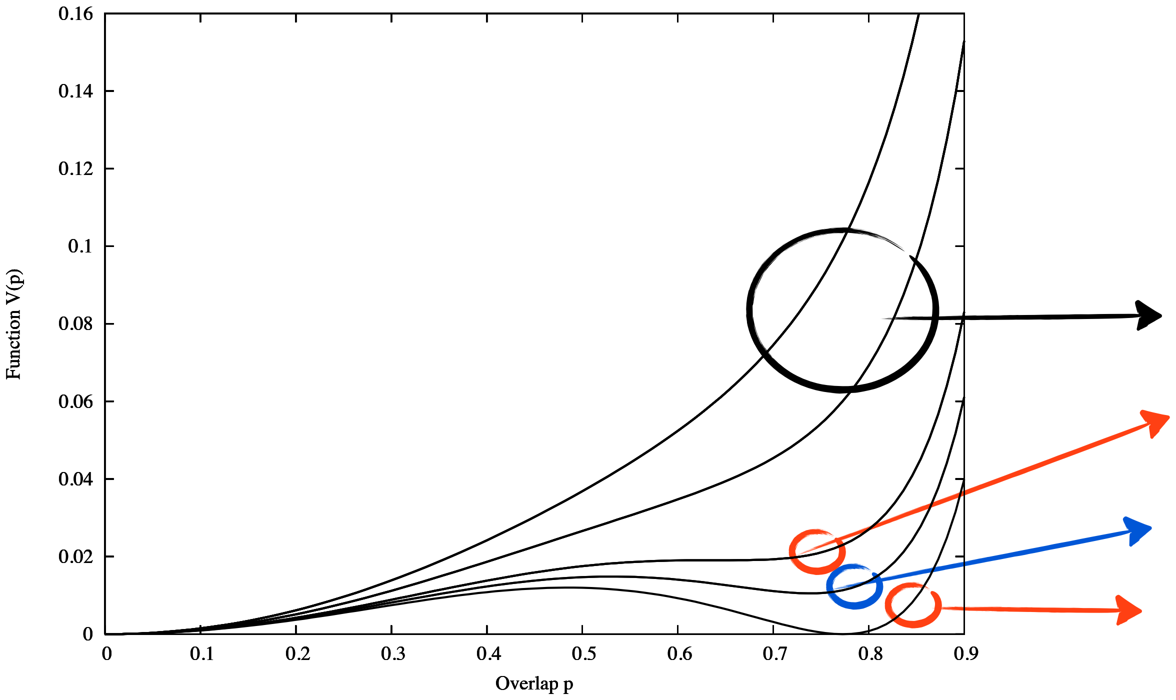

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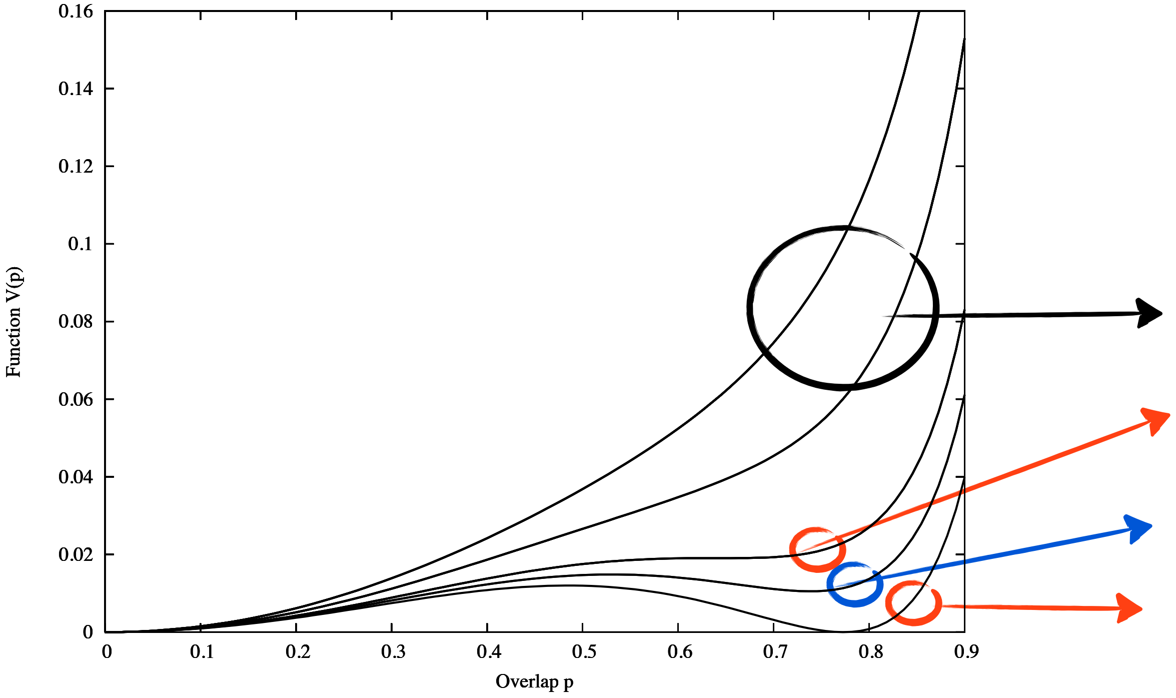

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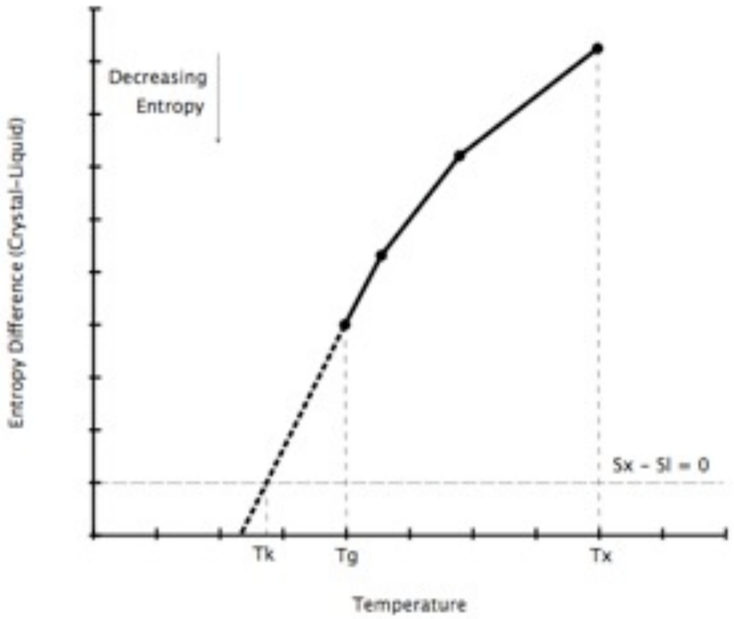

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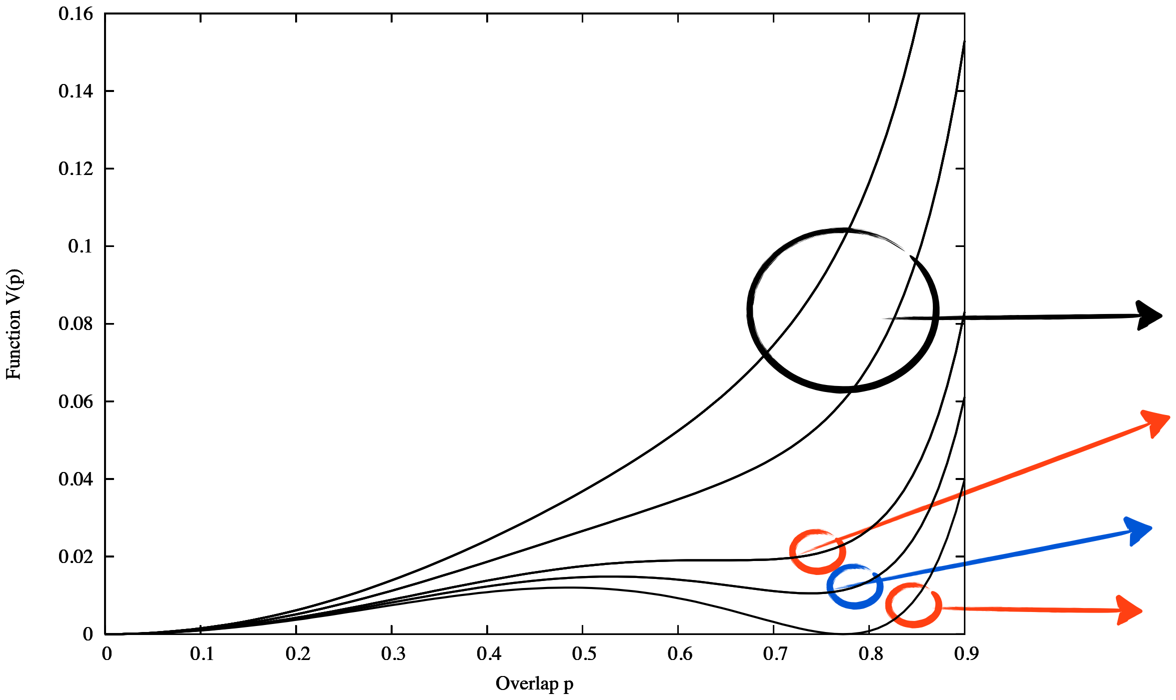

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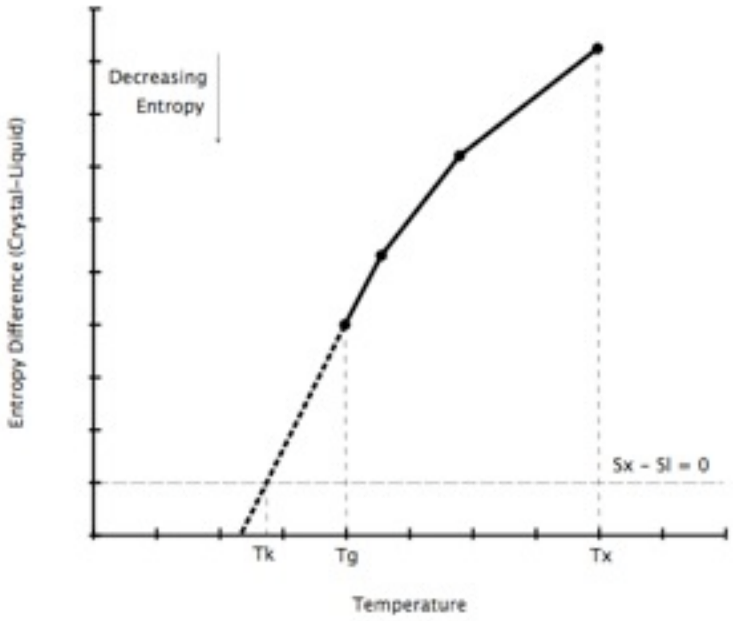
for $T \in [T_K, T_d]$

free energy

$$F(T) = f^*(T) - T\Sigma(f^*(T))$$

potential

$$V(p_{min}(T),T) - F(T) = T\Sigma(f^*(T))$$



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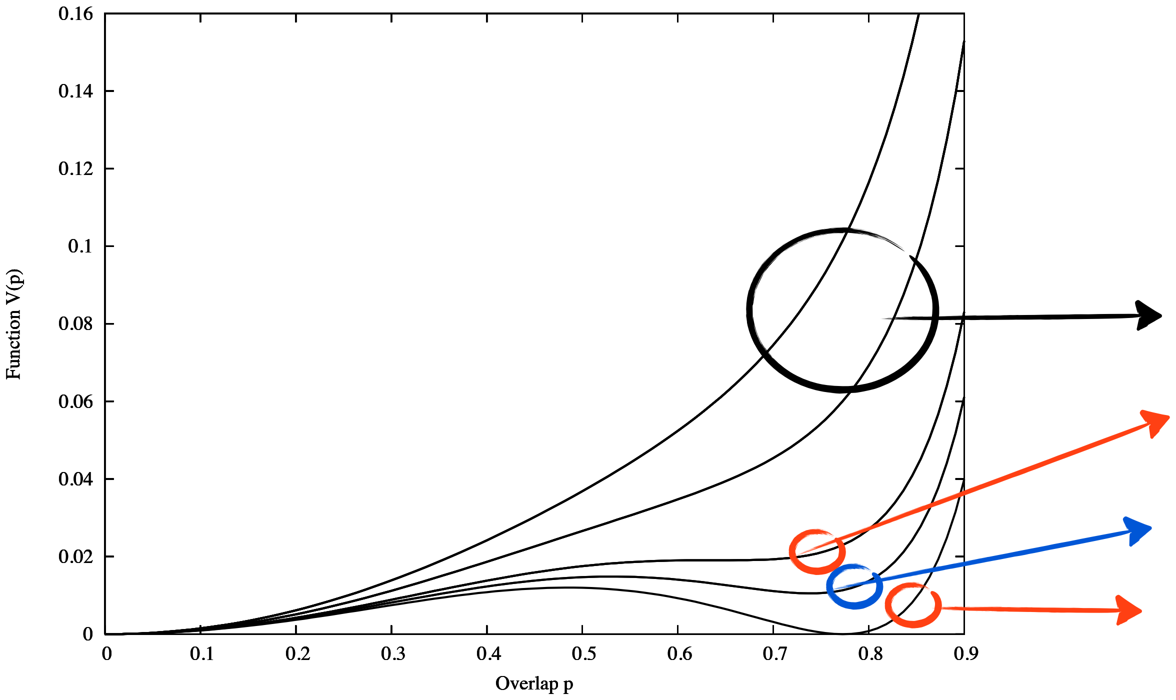

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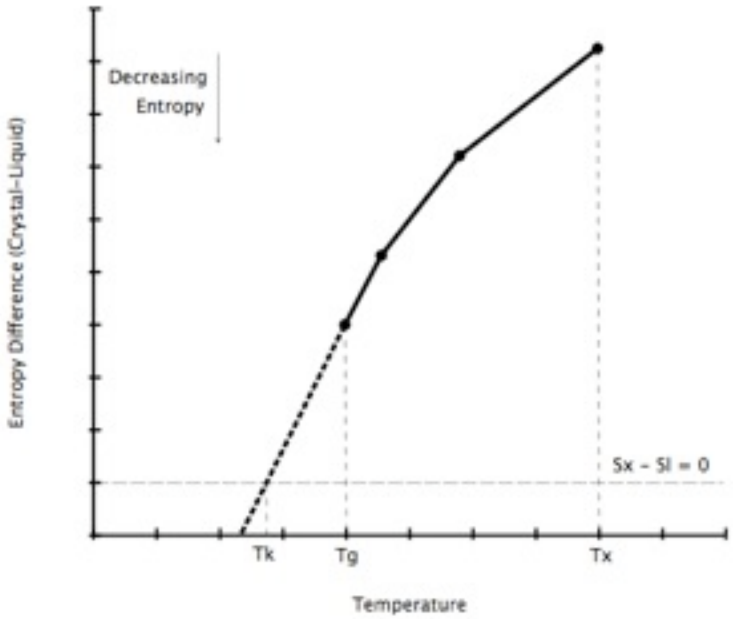

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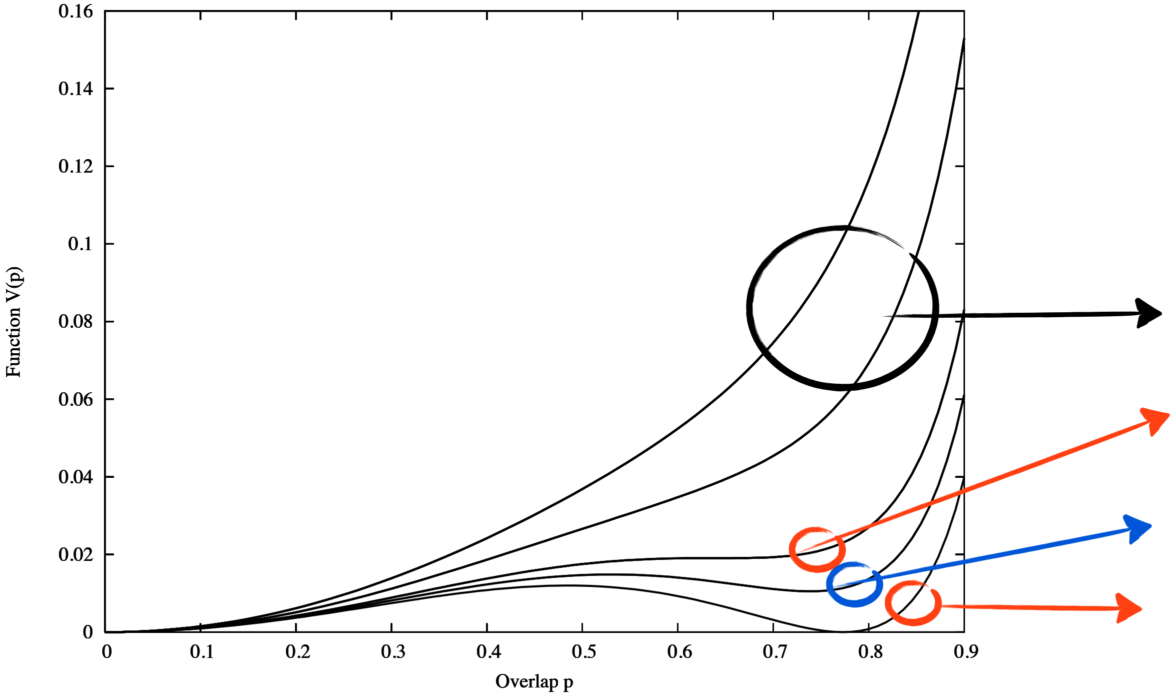

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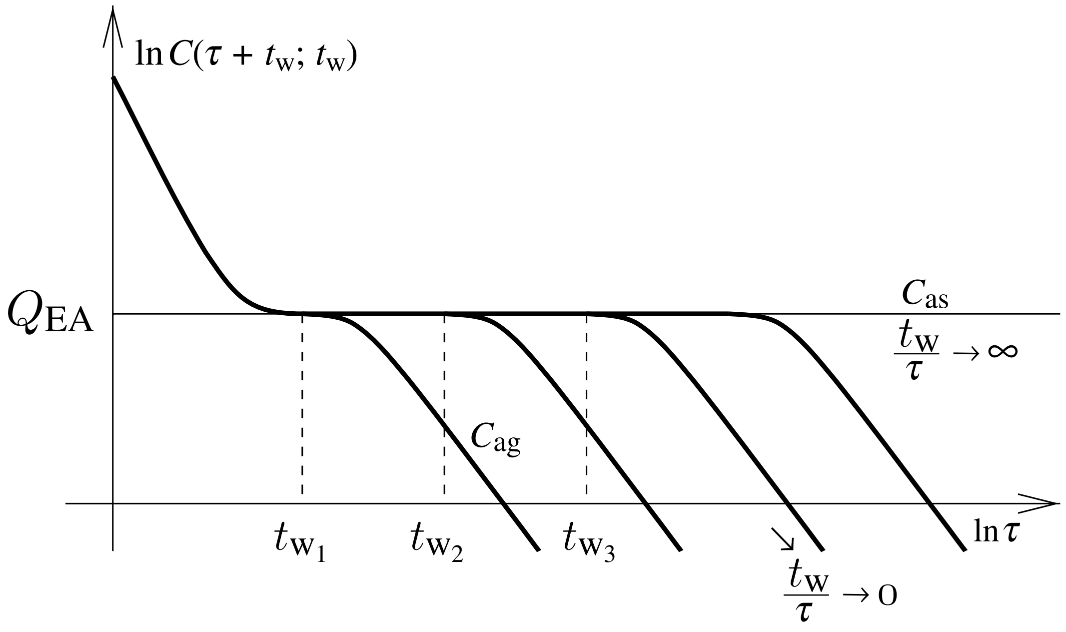

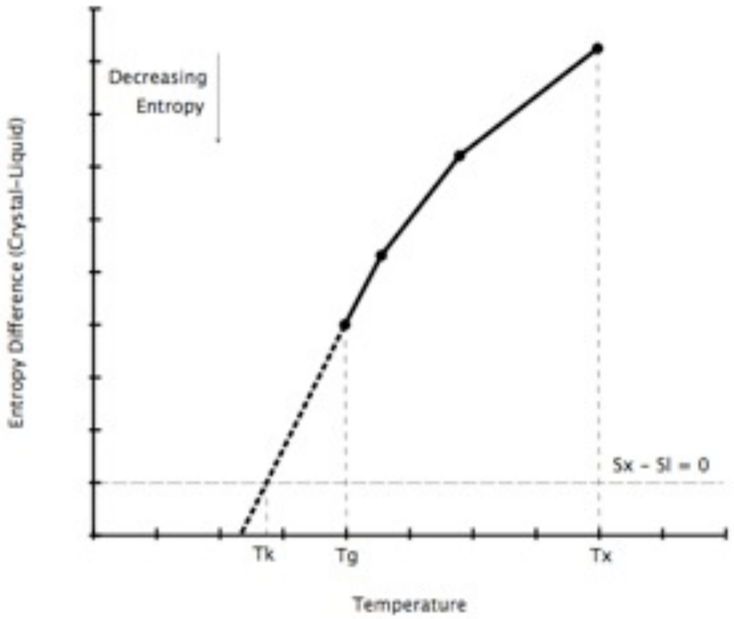

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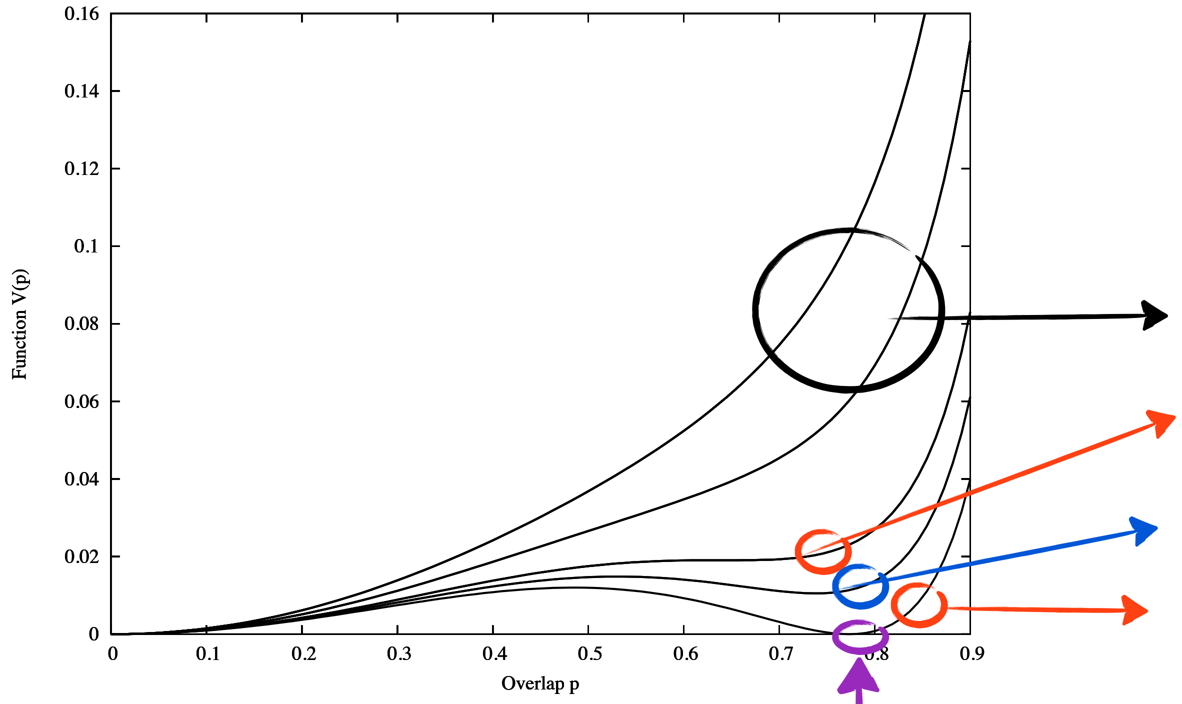

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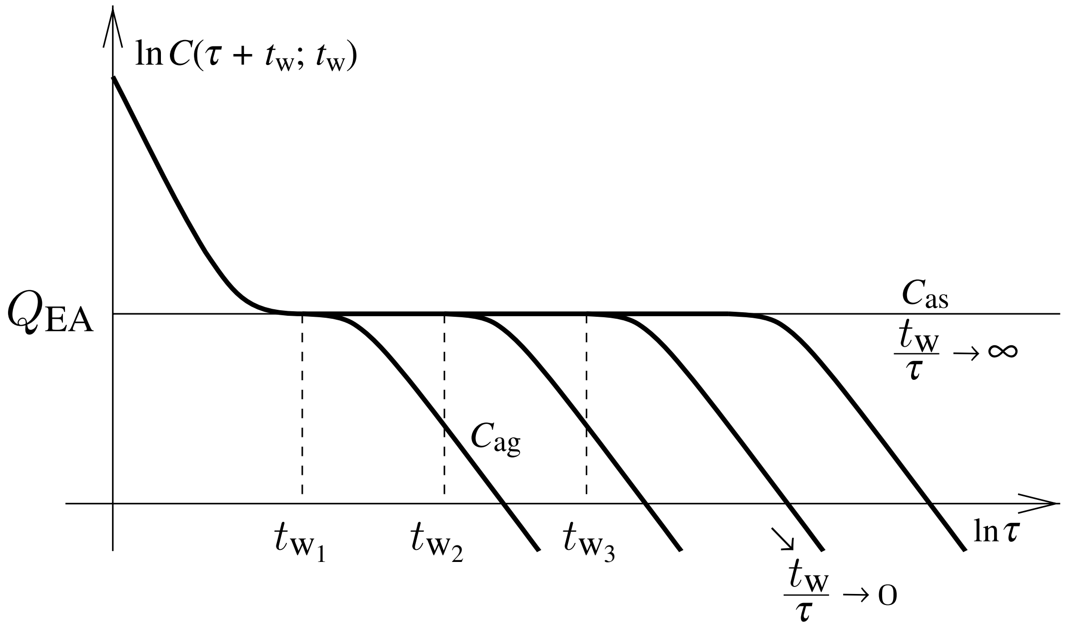

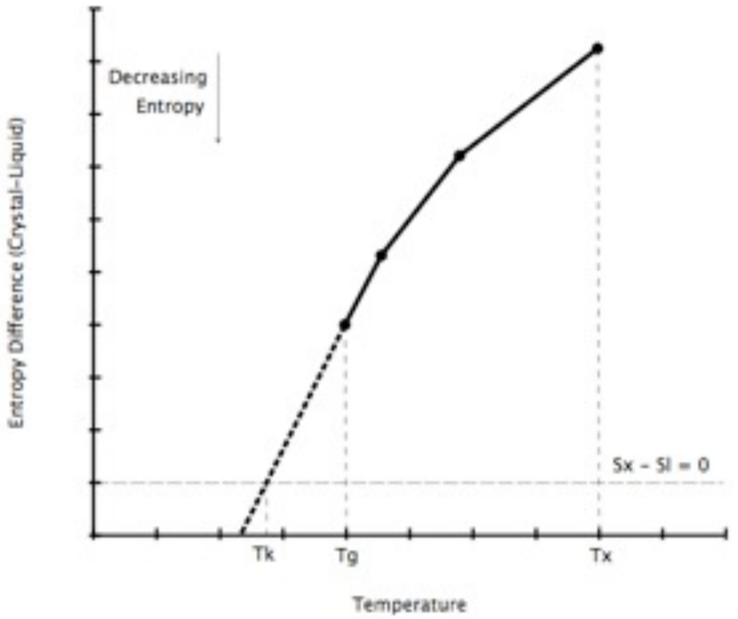

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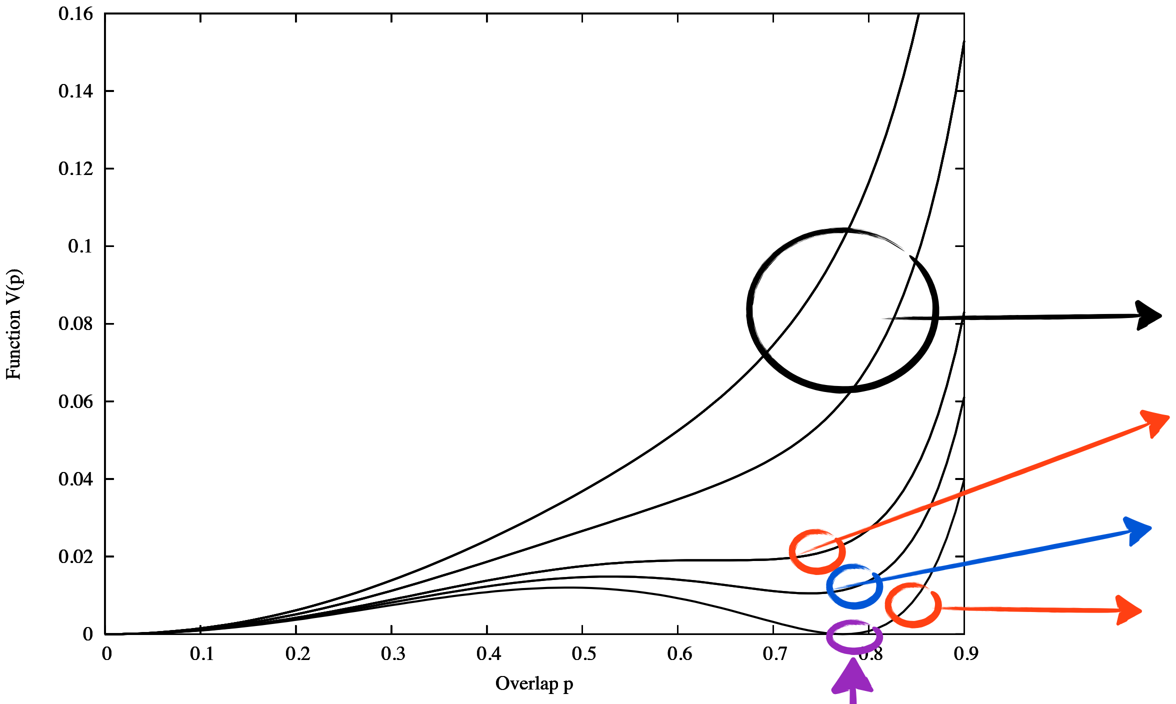

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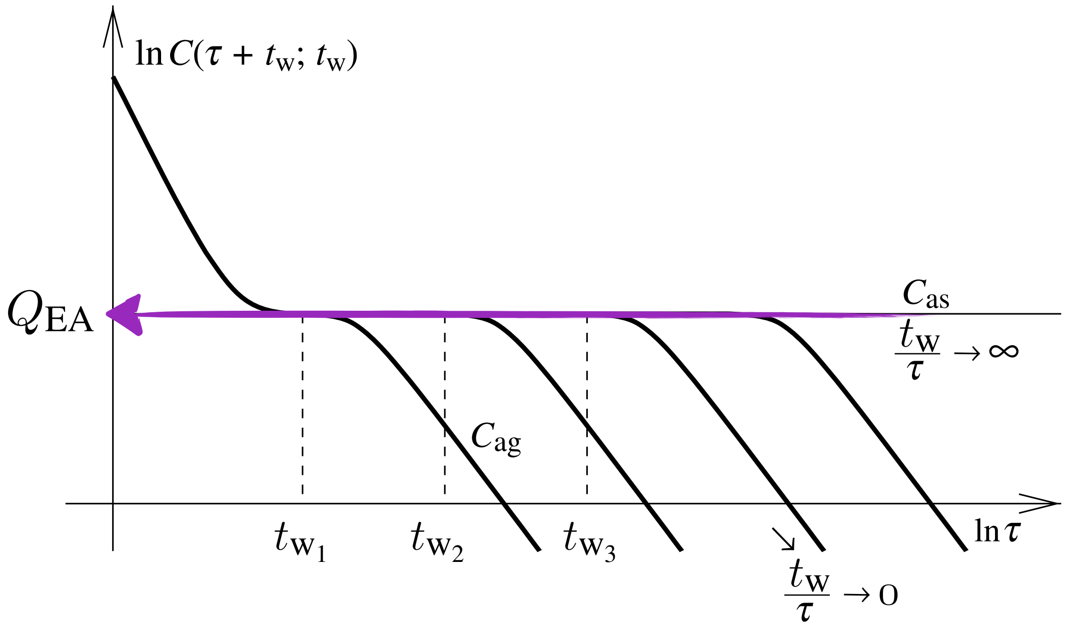

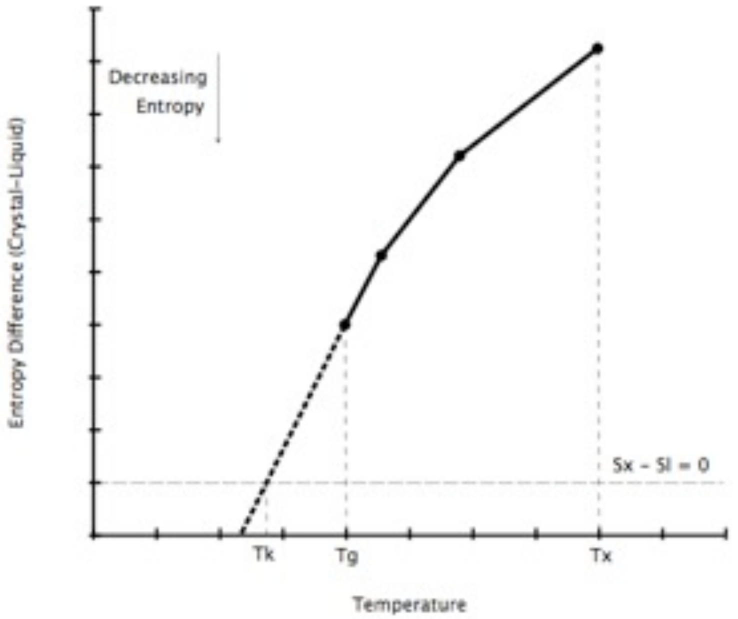

Average over disorder: Replica trick

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With the use of the replica trick

$$NV = -T\lim_{n\to 0}\lim_{m\to 0}\overline{\int d\underline{s}\exp\left(-\beta' H[\underline{s}]\right)Z[\beta']^{n-1}\Big(\frac{Z[\underline{s},\tilde{p}]^{m}-1}{m}\Big)}$$

with the constrained partition function $Z[\underline{s},\tilde{p}] = \int d\underline{\sigma}\,\mathrm{e}^{-\beta\mathrm{H}[\underline{\sigma}]}\,\delta(\tilde{\mathrm{p}}-\mathrm{Q}(\underline{\mathrm{s}},\underline{\sigma}))$

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Define the **'replicated partition function'**

$$Z^{(n,m)} = \int ds^{1}\mathrm{e}^{\beta' \mathrm{H}(\mathrm{s}^{1})}\mathrm{Z}(\beta')^{\mathrm{n}-1}\mathrm{Z}[\underline{\mathrm{s}},\tilde{\mathrm{p}}]^{\mathrm{m}} = \int \mathrm{ds}^{1}\mathrm{e}^{\beta' \mathrm{H}(\mathrm{s}^{1})}\mathrm{Z}(\beta')^{\mathrm{n}-1}\mathrm{e}^{\mathrm{m}\ln \mathrm{Z}[\underline{\mathrm{s}},\tilde{\mathrm{p}}]}$$

The potential can be recovered with

$$NV = -T\frac{\partial}{\partial m}\ln Z^{(n,m)}\Big|_{\substack{m=0\\ n=0}}$$

2+4 spin Hamiltonian: averaging over disorder

2+4 p-spin spherical **Hamiltonian**

$$H = -\sum_{i<j} J_{ij}\, s_i s_j - \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l \;, \qquad \sum_i s_i^2 = N$$

Replicated partition function

$$Z^{(n,m)} = \overline{\int Ds^a \int D\sigma^\alpha \exp\left[\beta' \sum_a^n H(s^a) + \beta \sum_\alpha^m H(\sigma^\alpha)\right] \underbrace{\prod_{\alpha=1}^m \delta\big(\sum_i s_i^1 \sigma_i^\alpha - N\tilde{p}\big)}_{\text{fixed distance between the two systems}}}$$

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Gaussian couplings

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Gaussian couplings

After averaging over the disorder

$$Z^{(n,m)} = \int Ds^a \int D\sigma^\alpha \prod_{i<j} \exp\left[\frac{p_2!}{4N^{p_2-1}}\Big(\beta_2 \sum_a^n s_i^a s_j^a + \beta \sum_\alpha^m \sigma_i^\alpha \sigma_j^\alpha\Big)^2\right]$$

$$\prod_{i<j<k<l} \exp\left[\frac{p_4!}{4N^{p_4-1}}\Big(\beta_4 \sum_a^n s_i^a s_j^a s_k^a s_l^a + \beta \sum_\alpha^m \sigma_i^\alpha \sigma_j^\alpha \sigma_k^\alpha \sigma_l^\alpha\Big)^2\right] \prod_{\alpha=1}^m \delta\big(\sum_i s_i^1 \sigma_i^\alpha - N\tilde{p}\big)$$

Introducing order parameters

Order parameter matrices

$$Q_{ab} = \frac{1}{N}\sum_i s_i^a s_i^b$$

$$R_{\alpha\beta} = \frac{1}{N}\sum_i \sigma_i^\alpha \sigma_i^\beta$$

$$P_{a\alpha} = \frac{1}{N}\sum_i s_i^a \sigma_i^\alpha$$

Single matrix

$$\mathbf{Q} = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix}$$

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Single matrix

$$\mathbf{Q} = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix}$$

Single spin vector

$$\begin{aligned} \underline{v} &= (v_1, v_2, \ldots, v_{n+m}) \\ &= (s_1, \ldots, s_n, \sigma_1, \ldots, \sigma_m) \end{aligned}$$

Introducing

$$1 = \int d\mathbf{Q}_{\gamma\eta}\, \delta\Big(N\mathbf{Q}_{\gamma\eta} - \sum_i v_i^\gamma v_i^\eta\Big)$$

Introducing order parameters

FPU problem

Potential Method

2+4 p-spin spherical Model

Potential Method

Disorder and replicas

Looking for minima

Hamiltonian dynamics

Generic equation of dynamics

Correlation and Response

Lagrangian multiplier

3=p-spin spherical Model

Potential Method

Correlation and Response

Future developments

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Introducing

$$1 = \int d\mathbf{Q}_{\gamma\eta}\, \delta\Big(N\mathbf{Q}_{\gamma\eta} - \sum_i v_i^\gamma v_i^\eta\Big)$$

We obtain

$$Z^{(n,m)} = \int Dv^\gamma \int D\mathbf{Q}_{\gamma\eta}\, \delta\Big(N\mathbf{Q}_{\gamma\eta} - \sum_i v_i^\gamma v_i^\eta\Big) \exp\Big[\frac{N}{4}\Big(\beta_2^2 \sum_{a,b}^{n} Q_{ab}^2 + 2\beta_2\beta \sum_{a,\alpha} P_{a,\alpha}^2 + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^2\Big)\Big]$$

$$\exp\Big[\frac{N}{4}\Big(\beta_4^2 \sum_{a,b}^{n} Q_{ab}^4 + 2\beta_4\beta \sum_{a,\alpha} P_{a,\alpha}^4 + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^4\Big)\Big] \prod_{\gamma=n+1}^{n+m} \delta\big(\sum_i v_i^1 v_i^\gamma - N\tilde{p}\big)$$

Generalized RS ansatz

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Using a **saddle point** technique to estimate the integral

$$Z^{(n,m)} = \int D\mathbf{Q}_{\gamma\eta} \int D\lambda_{\gamma\eta} \exp[-NS(\lambda,\mathbf{Q})] \simeq \exp[-NS(\lambda^*,\mathbf{Q}^*)]$$

$$\boxed{\begin{aligned}\frac{1}{N}\ln Z^{n,m} = {} & \frac{1}{4}\Big(\beta_2^2\sum_{a,b}^{n} Q_{ab}^2 + 2\beta_2\beta\sum_{a,\alpha}^{n,m} P_{a,\alpha}^2 + \beta^2\sum_{\alpha,\beta}^{m} R_{\alpha,\beta}^2\Big) \\ & + \frac{1}{4}\Big(\beta_4^2\sum_{a,b}^{n} Q_{ab}^4 + 2\beta_4\beta\sum_{a,\alpha}^{n,m} P_{a,\alpha}^4 + \beta^2\sum_{\alpha,\beta}^{m} R_{\alpha,\beta}^4\Big) + \frac{1}{2}\ln\det\begin{pmatrix} Q & P \\ P^T & R \end{pmatrix}\end{aligned}}$$

The Effective Potential can be obtained using

$$NV = -T\frac{\partial}{\partial m}\ln Z^{(n,m)}\Big|_{\substack{m=0\\ n=0}}$$

Generalized RS ansatz

Using a **saddle point** technique to estimate the integral

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Ansatz for the Overlap Matrices

$$\begin{aligned} Q_{ab} &= \delta_{ab} + (1-\delta_{ab})q \\ P_{a\alpha} &= \tilde{p}\,\delta_{\alpha n} + (1-\delta_{\alpha n})s \\ R_{\alpha\beta} &= \delta_{\alpha\beta} + (1-\delta_{\alpha\beta})r \end{aligned} \longrightarrow \mathbf{Q} = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix} = \left(\begin{array}{cccccccc} \multicolumn{4}{c}{\overbrace{\hphantom{1 \quad q \quad \cdots \quad q}}^{n}} & \multicolumn{4}{c}{\overbrace{\hphantom{s \quad s \quad \cdots \quad s}}^{m}} \\ 1 & q & \cdots & q & s & s & \cdots & s \\ q & 1 & \cdots & q & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots & s & s & \cdots & s \\ q & q & \cdots & 1 & \tilde{p} & \tilde{p} & \cdots & \tilde{p} \\ \\ s & \cdots & s & \tilde{p} & 1 & r & \cdots & r \\ \vdots & \ddots & \vdots & \vdots & r & 1 & \cdots & r \\ s & \cdots & s & \tilde{p} & \vdots & \vdots & \ddots & \vdots \\ s & \cdots & s & \tilde{p} & r & r & \cdots & 1 \end{array}\right)$$

Looking for minima

Potential Function

$$\begin{aligned}\beta V = & -\frac{1}{4}(2\beta^2 + 2\beta(\beta_2 p^2 + \beta_4 p^4) - \beta^2(r^2 + r^4) - 2\beta(\beta_2 s^2 + \beta_4 s^4)) \\ & -\frac{1}{2}(\frac{-p^2 + 2p^2 q + r - 2qr + q^2 r - 2pqs + s^2}{1 - 2q + q^2 - r + 2qr - q^2 r} + \ln[1 - r])\end{aligned}$$

Looking for minima

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Determine minima

$$\begin{aligned}\frac{\partial V(q,s,r,\tilde{p})}{\partial q} &= 0 \\ \frac{\partial V(q,s,r,\tilde{p})}{\partial s} &= 0 \\ \frac{\partial V(q,s,r,\tilde{p})}{\partial r} &= 0\end{aligned} \quad \longrightarrow \quad \frac{\partial V(q,s,r,\tilde{p})}{\partial \tilde{p}} = 0$$

Looking for minima

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Simplified case: only p = 4 spin and

$$\beta_2 = \beta_4 = \beta, \quad q = 0, \quad s = 0$$

$$\downarrow$$

Potential Function

$$\beta V = -\frac{1}{4}(\beta^2 + 2\beta^2 \tilde{p}^4 - \beta^2 r^4) - \frac{1}{2}(\frac{r - \tilde{p}^2}{1 - r} + \ln[1-r])$$

[Franz, S., & Parisi, G. (1995). Recipes for metastable states in spin glasses. *Journal de Physique I*, *5*(11), 1401-1415]

Looking for minima

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$$\frac{\partial V(q,s,r,\tilde{p})}{\partial q}=0$$

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$$\frac{\partial V(q,s,r,\tilde{p})}{\partial r}=0$$

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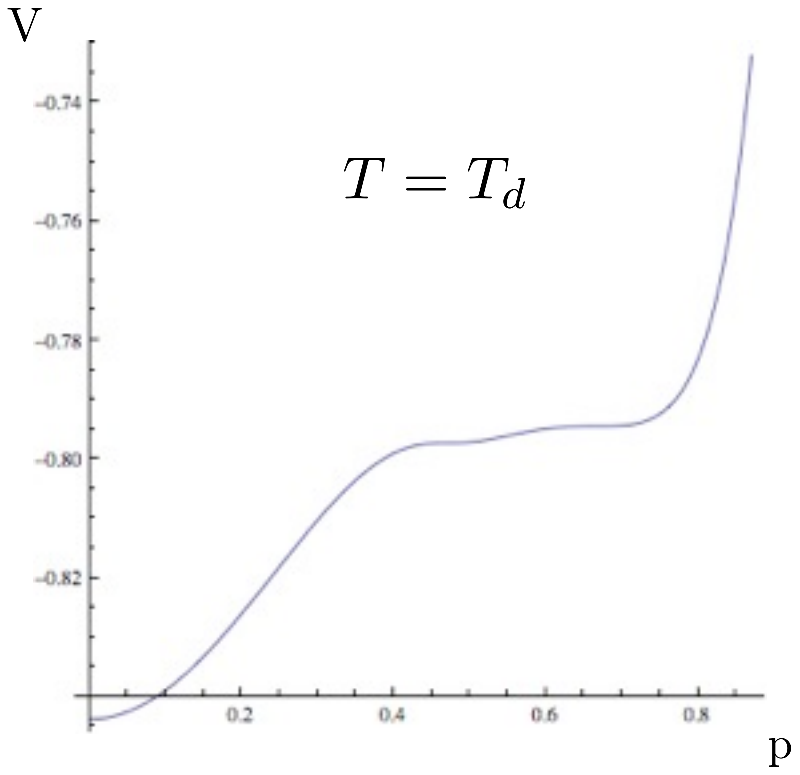

Looking for minima

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→

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↓

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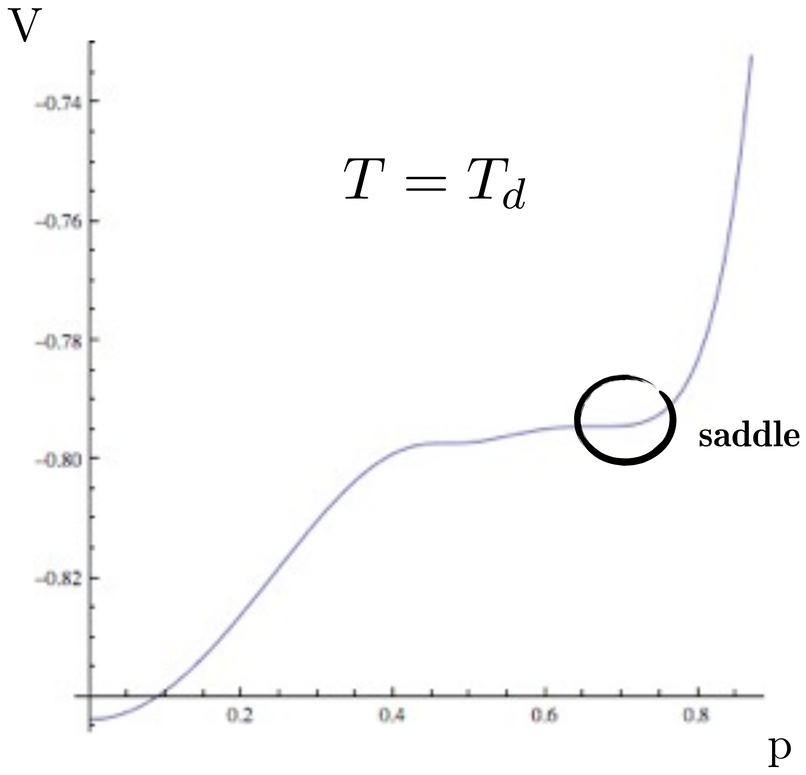

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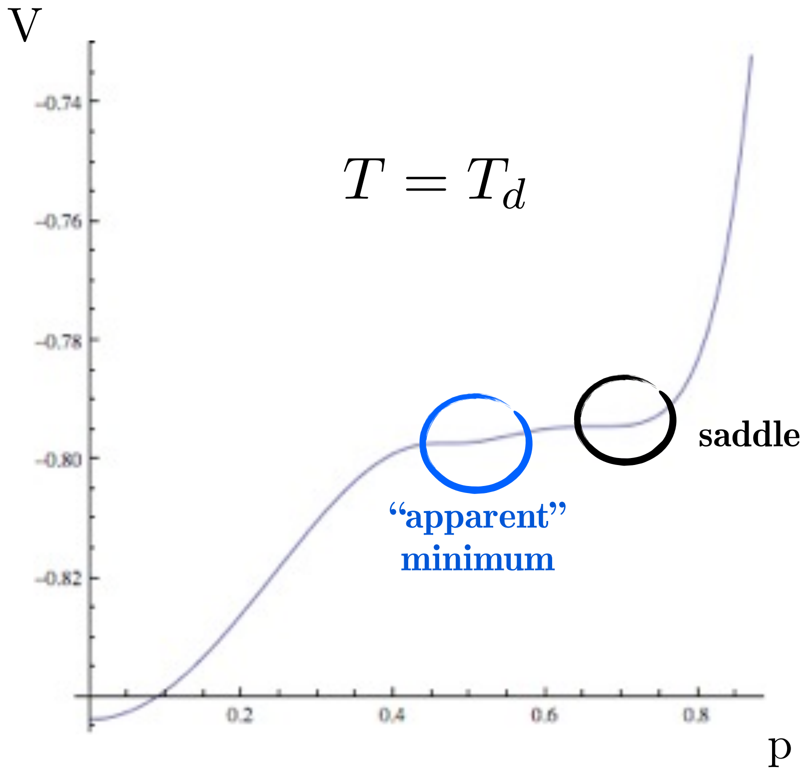
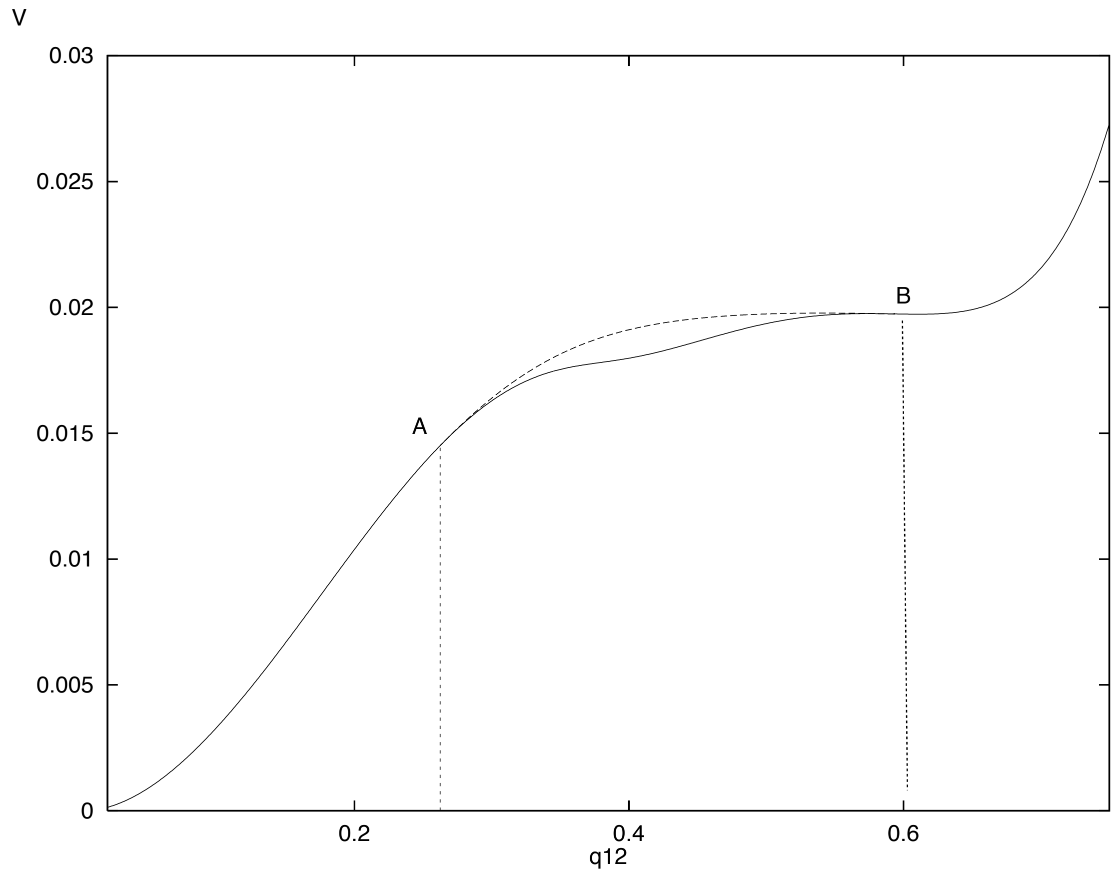


Potential: 1RSB corrections

1RSB treatment of the p>2 spin spherical model

Barrat, A., Franz, S., & Parisi, G. (1997). Temperature evolution and bifurcations of metastable states in mean-field spin glasses, with connections with structural glasses. *Journal of Physics A: Mathematical and General*, *30*(16), 5593.



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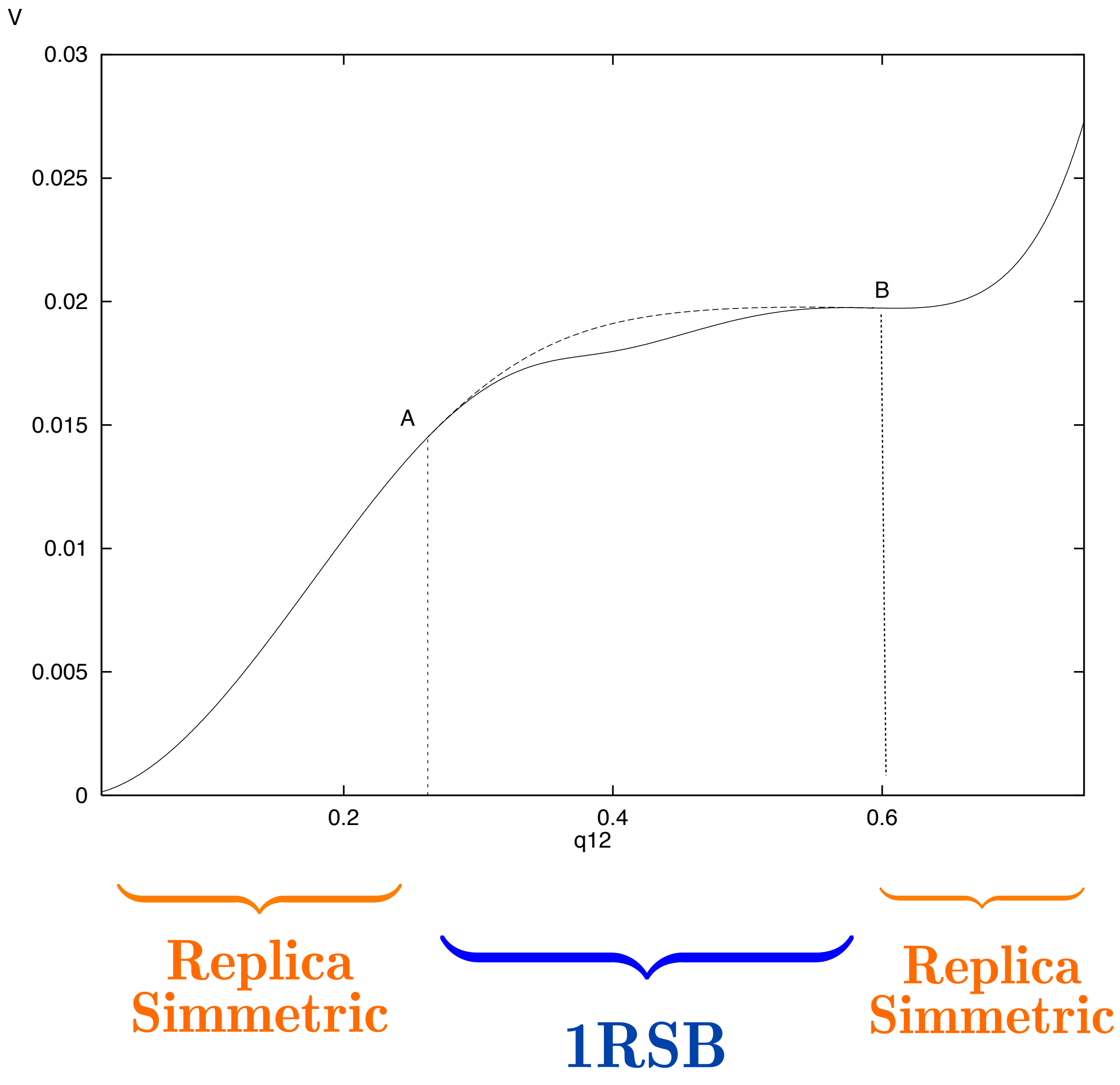

Potential: Results obtained so far

Hamiltonian

$$H = -\mu_2 \sum_{i<j} J_{ij}\, s_i s_j - \mu_4 \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l$$

p

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1-RSB

p

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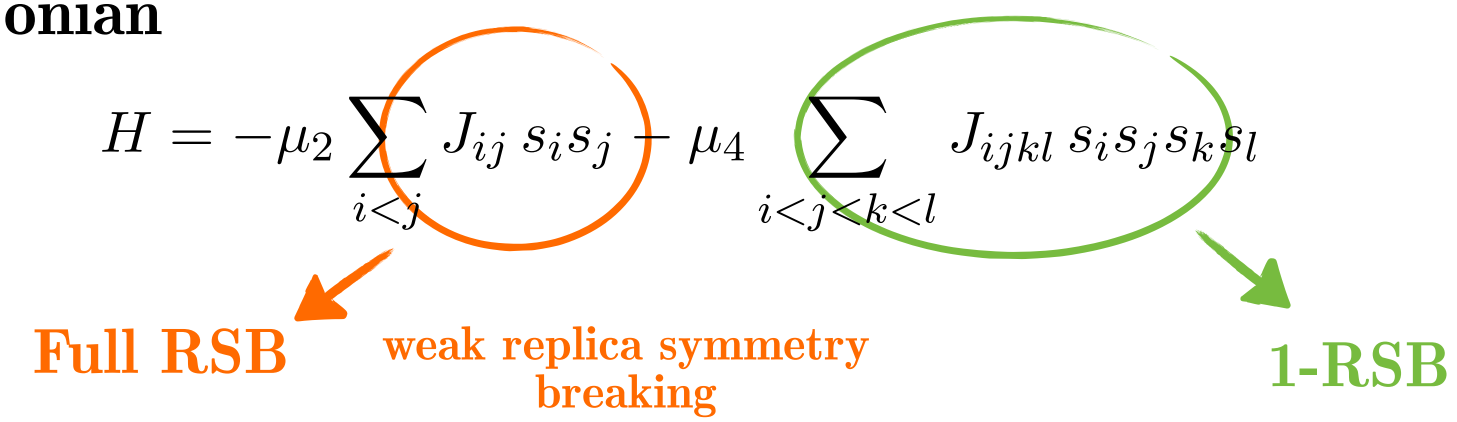

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Full RSB

weak replica symmetry breaking

1-RSB

p

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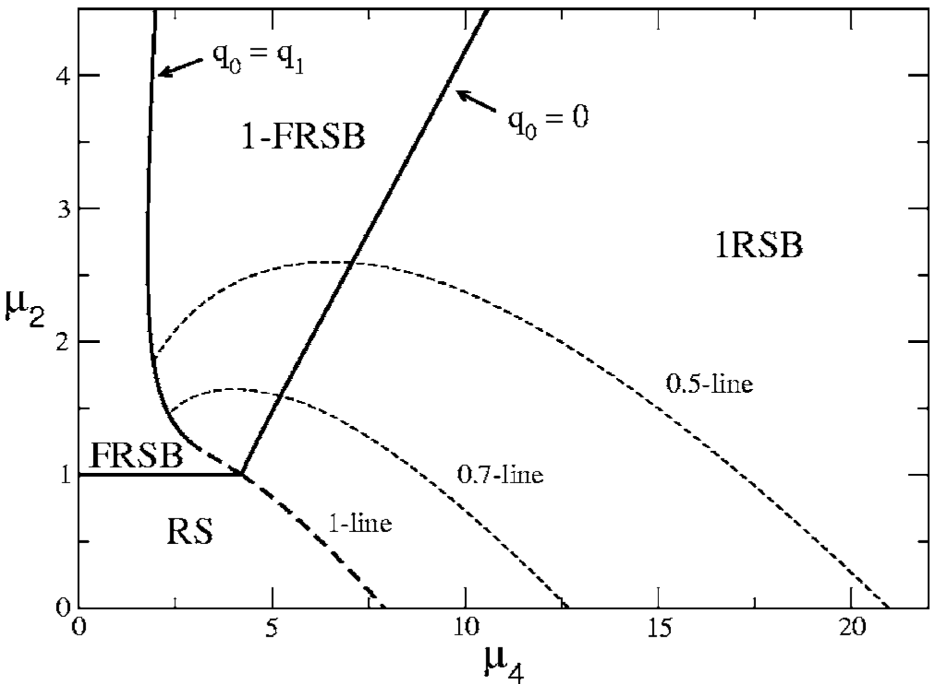
1-RSB

Phase diagram

The static phase diagram of the 2+4 model in the (μ_2, μ_4) plane



Crisanti, A., and L. Leuzzi. *"Spherical 2+ p spin-glass model: An exactly solvable model for glass to spin-glass transition." Physical review letters* 93.21 (2004): 217203.

p

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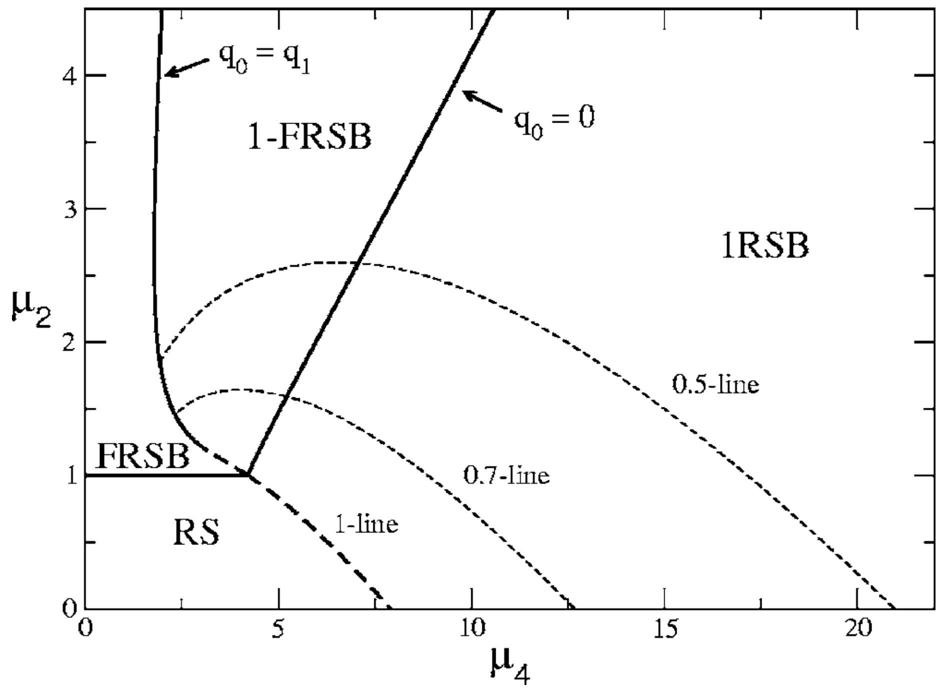

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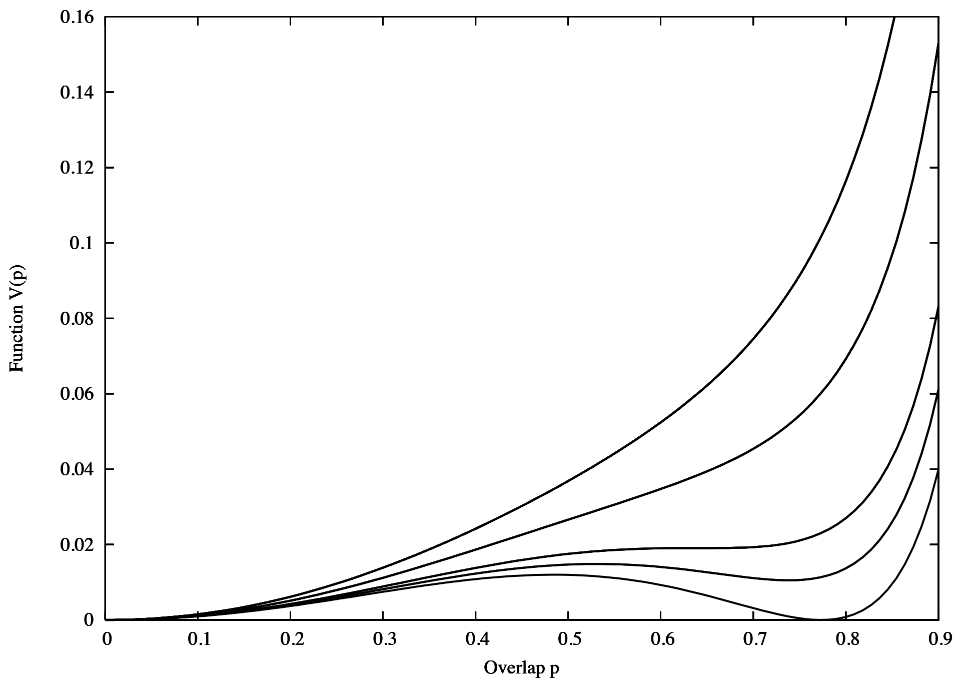
When $\gamma_2 \ll \gamma_4$



p

[S. Franz and G. Semerjian. *Analytical approaches to time and length scales in models of glasses* in *Dynamical heterogeneities in glasses, colloids and granular materials.* Oxford University Press, 2011.]

Potential: Results obtained so far

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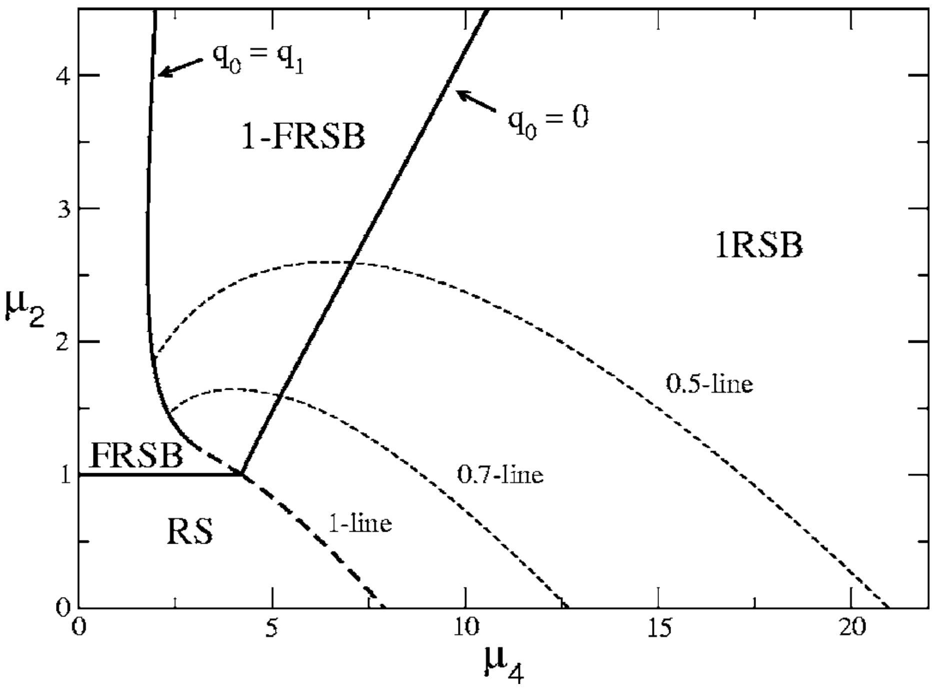

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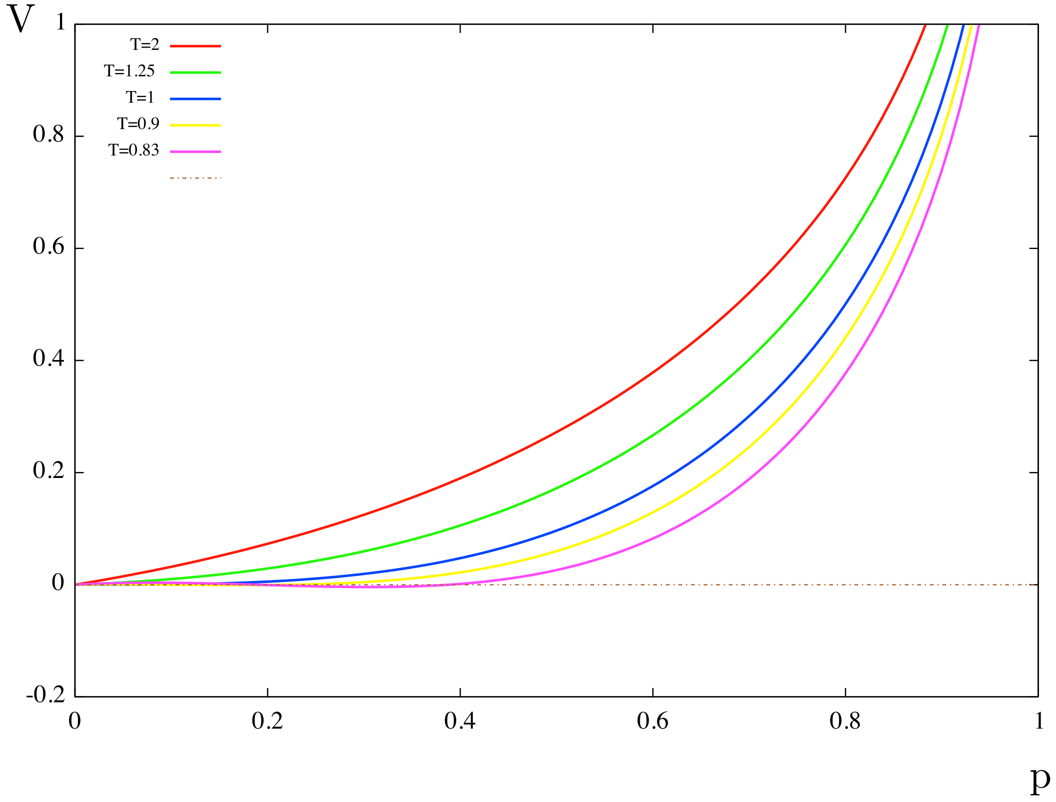
When $\gamma_2 \gg \gamma_4$



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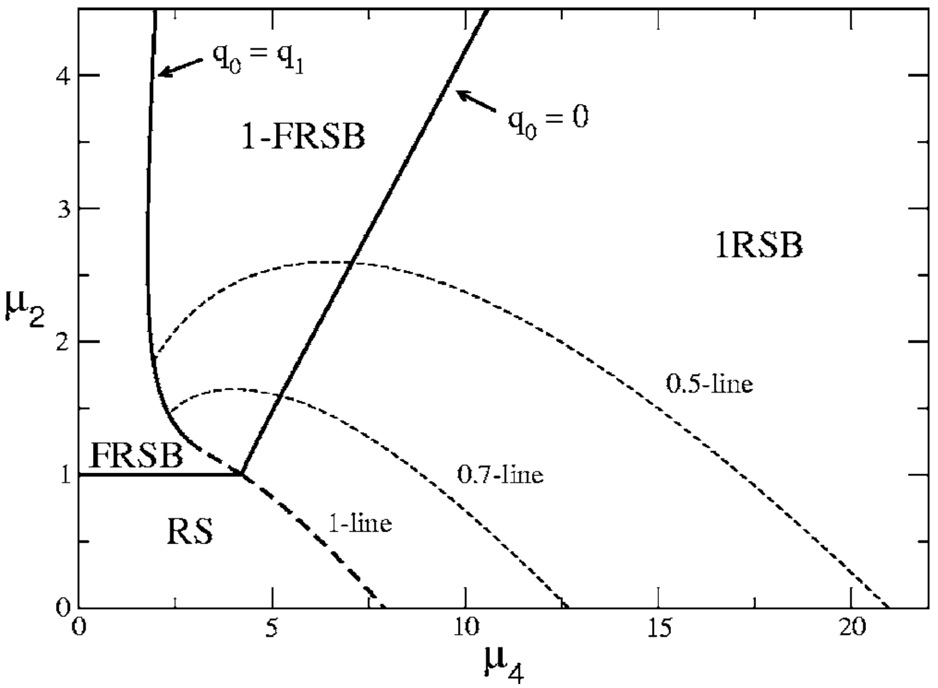

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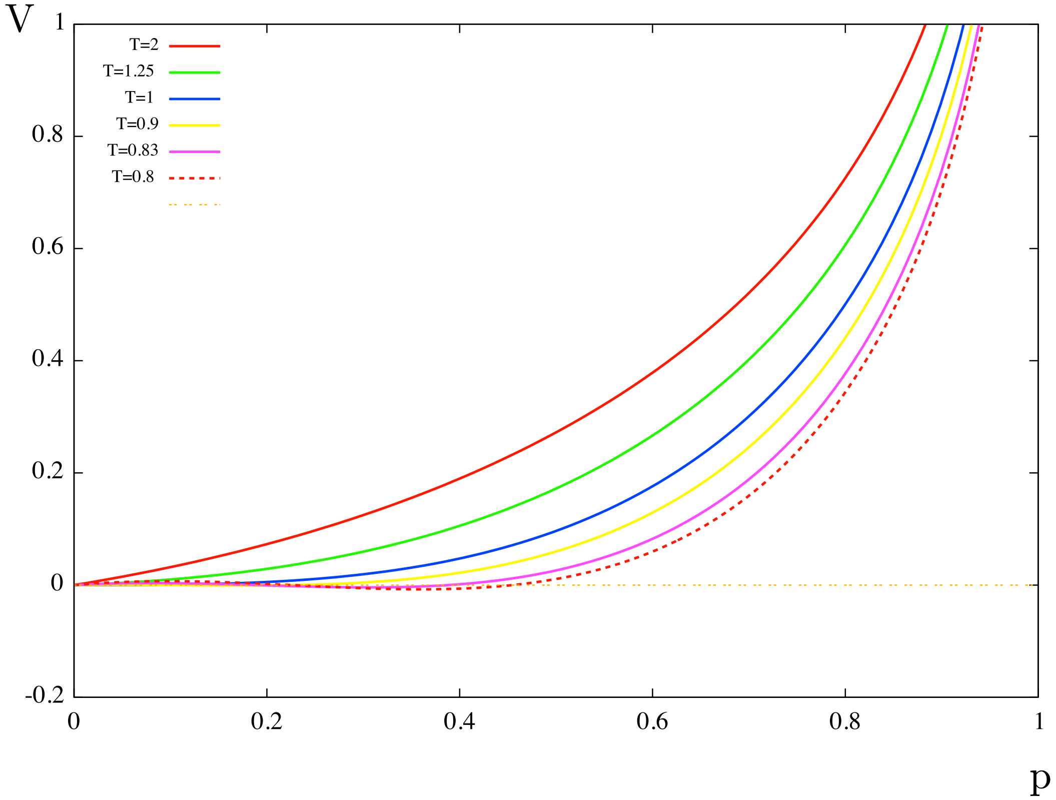

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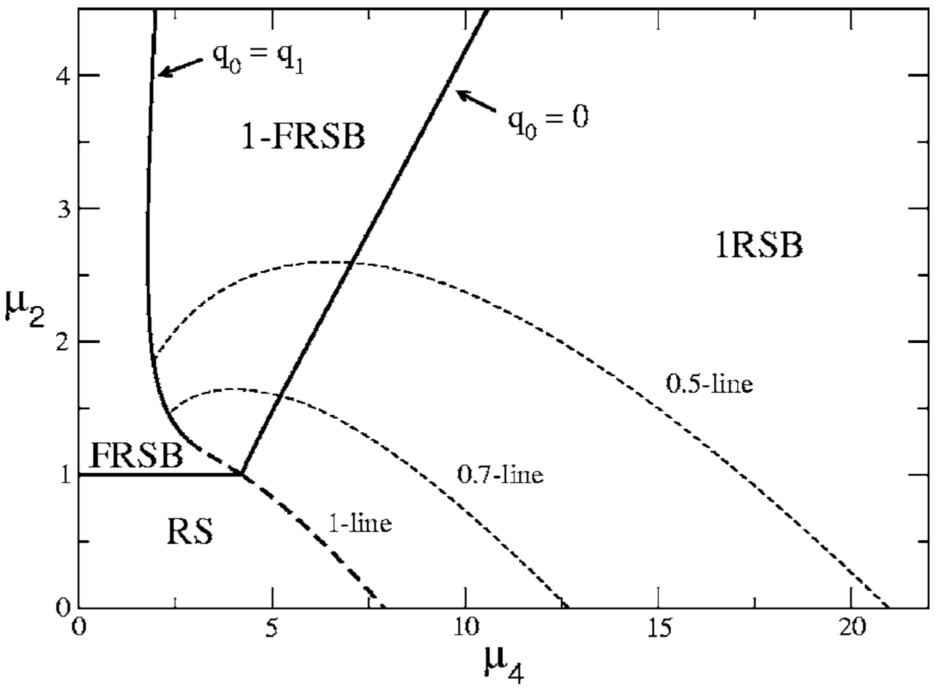
1-RSB

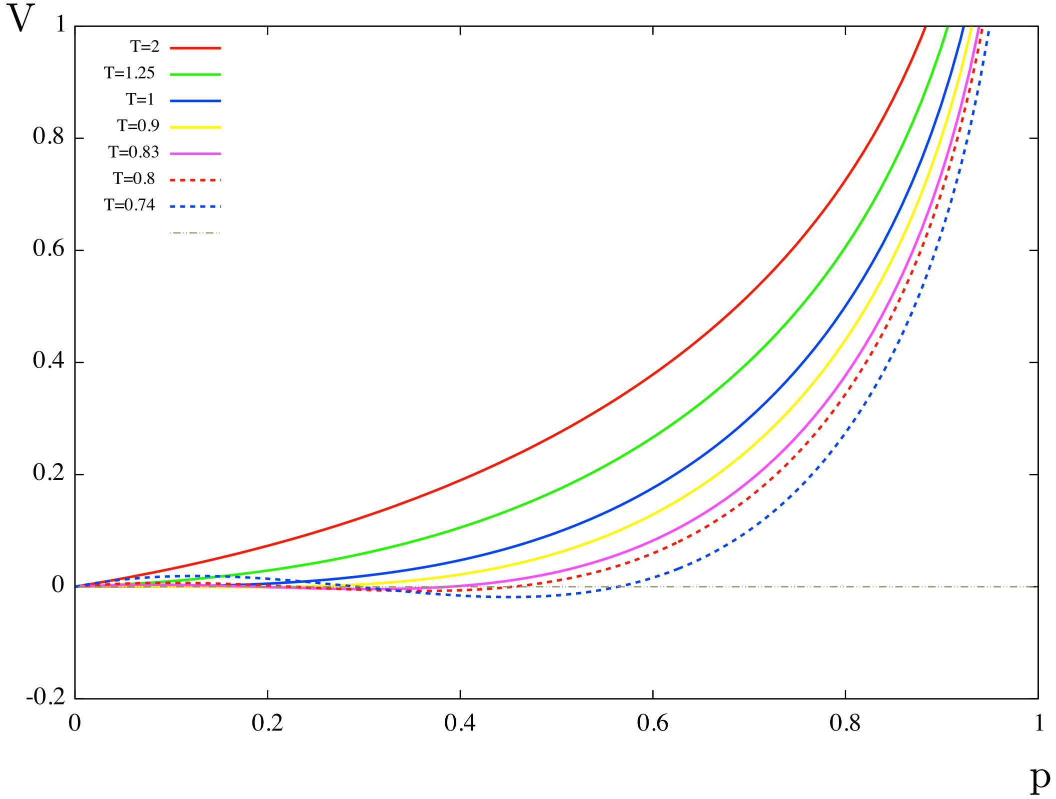
Phase diagram

The static phase diagram of the 2+4 model in the (μ_2, μ_4) plane



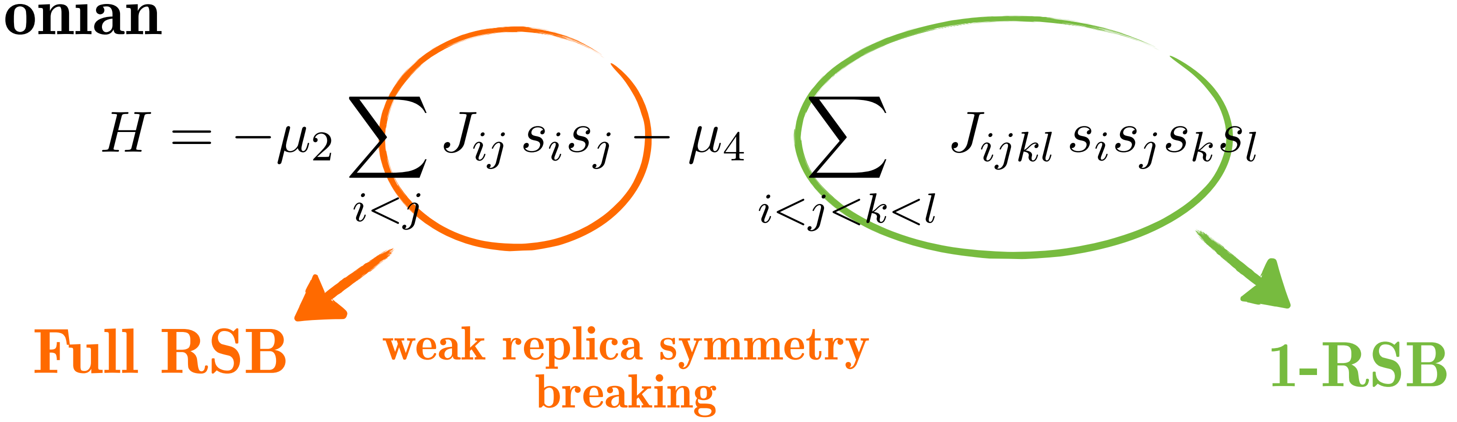
Crisanti, A., and L. Leuzzi. *"Spherical 2+ p spin-glass model: An exactly solvable model for glass to spin-glass transition."* *Physical review letters* 93.21 (2004): 217203.

When $\gamma_2 \gg \gamma_4$



Potential: Results obtained so far

Hamiltonian



$$H = -\mu_2 \sum_{i<j} J_{ij}\, s_i s_j - \mu_4 \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l$$

Full RSB

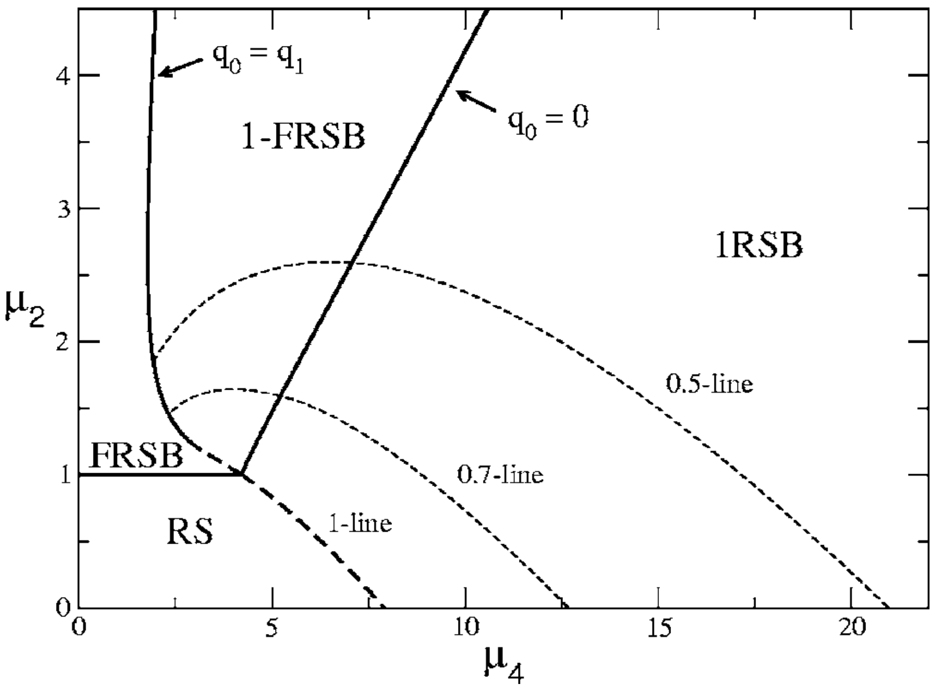
weak replica symmetry breaking

1-RSB

we need a Full RSB treatment

Phase diagram

The static phase diagram of the 2+4 model in the (μ_2, μ_4) plane



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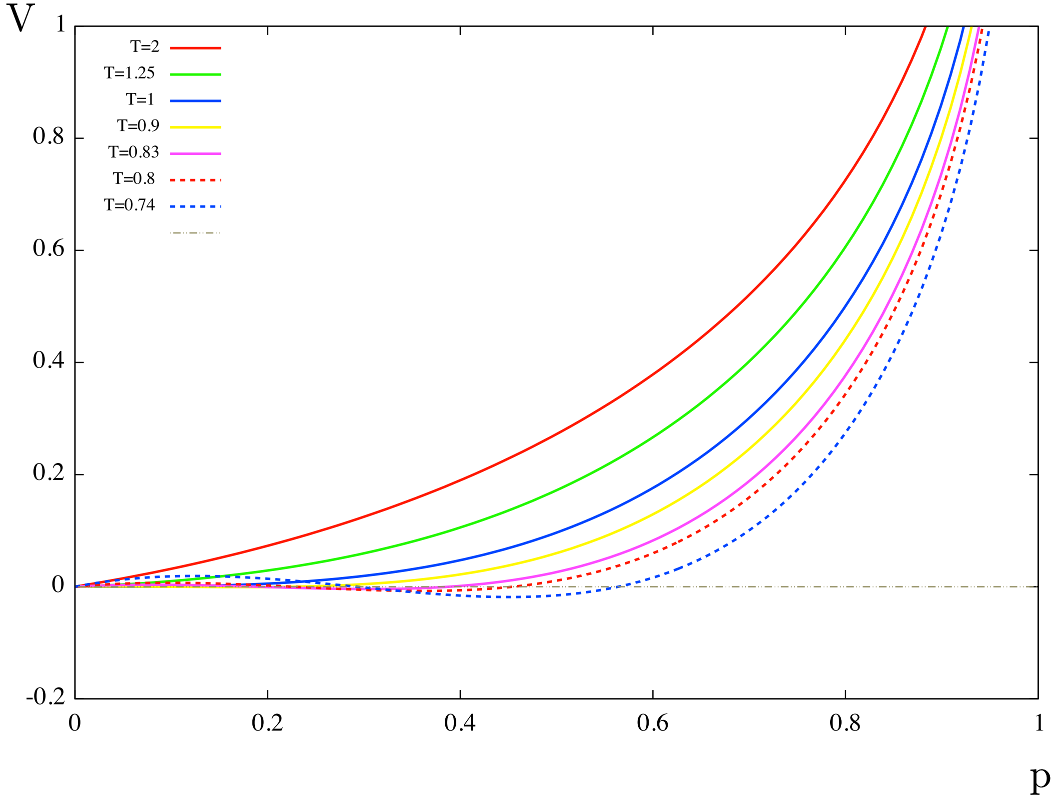
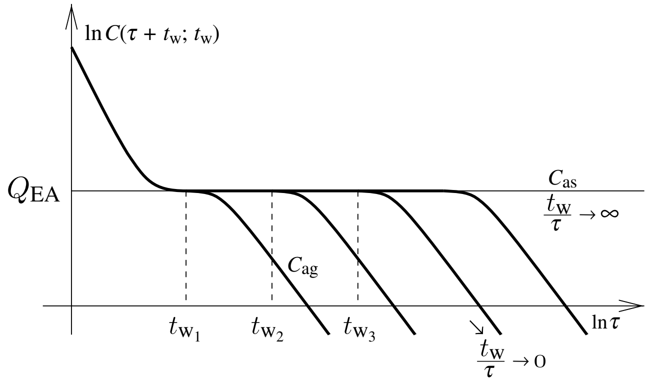


Hamiltonian Dynamics



Hamiltonian dynamics

Hamiltonian:

$$H = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \sum_i h_i s_i + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N)$$

with $$V_J(\underline{s}(t)) = -\sum_{i<j} J_{ij}\, s_i s_j - \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l$$

Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{s}_i = p_i$$

$$\frac{\partial H}{\partial s_i} = -\dot{p}_i = \frac{\partial V}{\partial s_i} + \mu_x s_i$$

Hamiltonian dynamics

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(Newtonian) **Equation of motion:** $\dot{p}_i = \ddot{s}_i = -\frac{\partial H}{\partial s_i} = -\frac{\partial V}{\partial s_i} - \mu_x s_i + h_i(t)$

which explicitly reads

$$\ddot{s}_i = -\mu_x(t) s_i(t) + \sum_j J_{ij}\, s_j(t) + \sum_{j<k<l} J_{ijkl}\, s_j(t) s_k(t) s_l(t) + h_i(t)$$

Hamiltonian dynamics

Hamiltonian:

$$H = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \sum_i h_i s_i + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N)$$

with $\quad V_J(\underline{s}(t)) = -\sum_{i<j} J_{ij}\, s_i s_j - \sum_{i<j<k<l} J_{ijkl}\, s_i s_j s_k s_l$

Hamilton's equations

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$$\boxed{\ddot{s}_i = -\mu_x(t) s_i(t) + \sum_j J_{ij}\, s_j(t) + \sum_{j<k<l} J_{ijkl}\, s_j(t) s_k(t) s_l(t) + h_i(t)}$$

Multiplying by an observable and averaging

$$\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = -\,\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}\langle s_j(t) A(s(t'))\rangle) + \sum_{j<k<l} \mathbb{E}(J_{ijkl}\,\langle s_j(t) s_k(t) s_l(t) A(s(t'))\rangle)$$

Martin-Siggia-Rose formalism

From the equation of motion

$$\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = -\,\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}\langle s_j(t) A(s(t'))\rangle) + \sum_{j<k<l} \mathbb{E}(J_{ijkl}\,\langle s_j(t) s_k(t) s_l(t) A(s(t'))\rangle)$$

Martin-Siggia-Rose formalism

From the equation of motion

$$\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = -\,\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}\langle s_j(t) A(s(t'))\rangle) + \sum_{j<k<l} \mathbb{E}(J_{ijkl}\,\langle s_j(t) s_k(t) s_l(t) A(s(t'))\rangle)$$

Taking averages $\langle \ldots \rangle$ → **Martin-Siggia-Rose formalism**

$$P[s]\mu(s(0)) = \int_{-\infty}^{\infty} \Big(\prod_{u=0}^{t} \frac{d\hat{s}_i(u)}{2\pi}\Big) \exp\Big\{\sum_i \int_0^t du \Big[i\hat{s}_i(u)\Big(-\ddot{s}_i(u) - \frac{\partial H_J}{\partial s_i(u)}\Big)\Big]\Big\}\mu(s(0))$$

$$= \int_{-\infty}^{\infty} \Big(\prod_{u=0}^{t} \frac{d\hat{s}_i(u)}{2\pi}\Big) \exp\Big\{\sum_i \int_0^t du \Big[i\hat{s}_i(u)\Big(-\ddot{s}_i(u) - \mu_x(u) s_i(u) + \sum_j J_{ij}\, s_j + \sum_{j<k<l} J_{ijkl}\, s_j s_k s_l + h_i(u)\Big)\Big]\Big\}\mu(s(0))$$

Martin-Siggia-Rose formalism

From the equation of motion

$$\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = -\,\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}\langle s_j(t) A(s(t'))\rangle) + \sum_{j<k<l} \mathbb{E}(J_{ijkl}\,\langle s_j(t) s_k(t) s_l(t) A(s(t'))\rangle)$$

Taking averages $\langle \dots \rangle$ → **Martin-Siggia-Rose formalism**

$$P[s]\mu(s(0)) = \int_{-\infty}^{\infty} \Big(\prod_{u=0}^{t} \frac{d\hat{s}_i(u)}{2\pi}\Big) \exp\Big\{\sum_i \int_0^t du\Big[i\hat{s}_i(u)\Big(-\ddot{s}_i(u) - \frac{\partial H_J}{\partial s_i(u)}\Big)\Big]\Big\}\mu(s(0))$$

$$= \int_{-\infty}^{\infty} \Big(\prod_{u=0}^{t} \frac{d\hat{s}_i(u)}{2\pi}\Big) \exp\Big\{\sum_i \int_0^t du\Big[i\hat{s}_i(u)\Big(-\ddot{s}_i(u) - \mu_x(u) s_i(u) + \sum_j J_{ij}\, s_j + \sum_{j<k<l} J_{ijkl}\, s_j s_k s_l + h_i(u)\Big)\Big]\Big\}\mu(s(0))$$

Let us observe

$$\frac{\partial}{\partial h_i(u)}\langle B(s(t))\rangle = \langle B(s(t))\, i\hat{s}_i(u)\rangle$$

Define **Correlation** and **Response**

$$\boxed{\begin{aligned} C(t,t') &= \frac{1}{N}\sum_i s_i(t) s_i(t') \to \langle s_i(t) s_i(t')\rangle \\ R(t,t') &= \frac{1}{N}\sum_i s_i(t)\, i\hat{s}_i(t') \to \langle s_i(t)\, i\hat{s}_i(t')\rangle = \frac{\partial\langle s_i(t)\rangle}{\partial h_j(t')} \end{aligned}}$$

Dynamics for a generic observable

Main equation of dynamics

$$
\begin{aligned}
\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = & -\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}^2)\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j A(s(t')) s_j(t)\rangle \\
& + \int_0^t du\,\mathbb{E}\langle s_i\, i\hat{s}_j A(s(t')) s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) A(s(t')) s_j(t)\rangle\Big] \\
& + \sum_{j<k<l} \mathbb{E}(J_{ijkl}^2)\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A(s(t')) s_j s_k s_l\rangle \\
& + \int_0^t du\,\mathbb{E}\langle s_i\,(i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A(s(t')) s_j s_k s_l\rangle \\
& + \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) A(s(t'))\, s_j s_k s_l\rangle\Big]
\end{aligned}
$$

Dynamics for a generic observable

Main equation of dynamics

$$\begin{aligned}\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = & -\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}^2)\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j A(s(t')) s_j(t)\rangle \\ & + \int_0^t du\,\mathbb{E}\langle s_i\, i\hat{s}_j A(s(t')) s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) A(s(t')) s_j(t)\rangle\Big] \\ & + \sum_{j<k<l} \mathbb{E}(J_{ijkl}^2)\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A(s(t')) s_j s_k s_l\rangle \\ & + \int_0^t du\,\mathbb{E}\langle s_i\,(i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A(s(t')) s_j s_k s_l\rangle \\ & + \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) A(s(t'))\, s_j s_k s_l\rangle\Big]\end{aligned}$$

Dynamics for a generic observable

Main equation of dynamics

$$\boxed{\begin{aligned}\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = &-\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \underline{\mathbb{E}(J_{ij}^2)}\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j A(s(t')) s_j(t)\rangle \\ &+ \int_0^t du\, \mathbb{E}\langle s_i\, i\hat{s}_j A(s(t')) s_j\rangle + \mathbb{E}\langle \beta_2(s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) A(s(t')) s_j(t)\rangle\Big] \\ &+ \sum_{j<k<l} \underline{\mathbb{E}(J_{ijkl}^2)}\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A(s(t')) s_j s_k s_l\rangle \\ &+ \int_0^t du\, \mathbb{E}\langle s_i\, (i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A(s(t')) s_j s_k s_l\rangle \\ &+ \mathbb{E}\langle \beta_4(s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) A(s(t'))\, s_j s_k s_l\rangle\Big]\end{aligned}}$$

Second moments

$$\longrightarrow \qquad \mathbb{E}(J_{i_1,\ldots,i_p}^2) = \frac{p!}{2N^{p-1}}$$

Dynamics for a generic observable

Main equation of dynamics

$$\boxed{\begin{aligned}\mathbb{E}\langle \ddot{s}_i \underline{A(s(t'))}\rangle = &-\mathbb{E}\langle \mu_x(t) s_i(t) \underline{A(s(t'))}\rangle + \sum_j \underline{\mathbb{E}(J_{ij}^2)}\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j \underline{A(s(t'))} s_j(t)\rangle \\ &+ \int_0^t du\,\mathbb{E}\langle s_i\, i\hat{s}_j \underline{A(s(t'))} s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) \underline{A(s(t'))} s_j(t)\rangle\Big] \\ &+ \sum_{j<k<l} \underline{\mathbb{E}(J_{ijkl}^2)}\Big[\int_0^t du\,\mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l \underline{A(s(t'))} s_j s_k s_l\rangle \\ &+ \int_0^t du\,\mathbb{E}\langle s_i\,(i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) \underline{A(s(t'))} s_j s_k s_l\rangle \\ &+ \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) \underline{A(s(t'))}\, s_j s_k s_l\rangle\Big]\end{aligned}}$$

Second moments

→ $$\mathbb{E}(J^2_{i_1,\dots,i_p}) = \frac{p!}{2N^{p-1}}$$

Dynamics for a generic observable

Main equation of dynamics

$$\boxed{\begin{aligned}
\mathbb{E}\langle \ddot{s}_i \underline{A(s(t'))}\rangle = &-\mathbb{E}\langle \mu_x(t) s_i(t) \underline{A(s(t'))}\rangle + \sum_j \underline{\mathbb{E}(J_{ij}^2)}\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j \underline{A(s(t'))} s_j(t)\rangle \\
&+ \int_0^t du\, \mathbb{E}\langle s_i\, i\hat{s}_j \underline{A(s(t'))} s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) \underline{A(s(t'))} s_j(t)\rangle\Big] \\
&+ \sum_{j<k<l} \underline{\mathbb{E}(J_{ijkl}^2)}\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l \underline{A(s(t'))} s_j s_k s_l\rangle \\
&+ \int_0^t du\, \mathbb{E}\langle s_i\, (i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) \underline{A(s(t'))} s_j s_k s_l\rangle \\
&+ \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) \underline{A(s(t'))}\, s_j s_k s_l\rangle\Big]
\end{aligned}}$$

Second moments

$$\longrightarrow \quad \mathbb{E}(J_{i_1,\ldots,i_p}^2) = \frac{p!}{2N^{p-1}}$$

Get equations for **Correlation** and **Response**

$$\longrightarrow \quad A(s(t')) = s_i(t') \longrightarrow C(t,t') = \frac{1}{N}\sum_i s_i(t) s_i(t') \to \langle s_i(t) s_i(t')\rangle$$

$$\longrightarrow \quad A(s(t')) = i\,\hat{s}_i(t') \longrightarrow R(t,t') = \frac{1}{N}\sum_i s_i(t)\, i\hat{s}_i(t') \to \langle s_i(t)\, i\hat{s}_i(t')\rangle = \frac{\partial \langle s_i(t)\rangle}{\partial h_j(t')}$$

Equations for Correlation and Response

FPU problem
Potential Method
2+4 p-spin spherical Model
Potential Method
Disorder and replicas
Looking for minima
Hamiltonian dynamics
Generic equation of dynamics
Correlation and Response
Lagrangian multiplier
3=p-spin spherical Model
Potential Method
Correlation and Response
Future developments

$$
\begin{aligned}
\frac{\partial^2 C(t,t')}{\partial t^2} &= -\mu_x(t) C(t,t') \\
&+ \frac{p_2}{2}\int_0^{t'} du R(t',u) C(t,u)^{p_2-1} + \frac{p_2(p_2-1)}{2}\int_0^t du\, C(t',u) R(t,u) C(t,u)^{p_2-2} \\
&+ \beta_2 \frac{p_2}{2}\Big(C(t',0) C(t,0)^{p_2-1} - K(0,t') K(0,t)^{p_2-1} \Big) \\
&+ \frac{p_4}{2}\int_0^{t'} du R(t',u) C(t,u)^{p_4-1} + \frac{p_4(p_4-1)}{2}\int_0^t du\, C(t',u) R(t,u) C(t,u)^{p_4-2} \\
&+ \beta_4 \frac{p_4}{2}\Big(C(t',0) C(t,0)^{p_4-1} - K(0,t') K(0,t)^{p_4-1} \Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 R(t,t')}{\partial t^2} &= -\,\mu_x(t) R(t,t') + \frac{p_2(p_2-1)}{2}\int_{t'}^t du\, R(u,t') R(t,u) C(t,u)^{p_2-2} \\
&+ \frac{p_4(p_4-1)}{2}\int_{t'}^t du\, R(u,t') R(t,u) C(t,u)^{p_4-2}
\end{aligned}
$$

Equations for Correlation and Response

$$\begin{aligned}\frac{\partial^2 C(t,t')}{\partial t^2} &= -\mu_x(t)C(t,t') \\ &+ \frac{p_2}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_2-1} + \frac{p_2(p_2-1)}{2}\int_0^t du\, C(t',u)R(t,u)C(t,u)^{p_2-2} \\ &+ \beta_2 \frac{p_2}{2}\Big(C(t',0)C(t,0)^{p_2-1} - \underline{K(0,t')K(0,t)^{p_2-1}}\Big) \\ &+ \frac{p_4}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_4-1} + \frac{p_4(p_4-1)}{2}\int_0^t du\, C(t',u)R(t,u)C(t,u)^{p_4-2} \\ &+ \beta_4 \frac{p_4}{2}\Big(C(t',0)C(t,0)^{p_4-1} - \underline{K(0,t')K(0,t)^{p_4-1}}\Big)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 R(t,t')}{\partial t^2} &= -\,\mu_x(t)R(t,t') + \frac{p_2(p_2-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p_2-2} \\ &+ \frac{p_4(p_4-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p_4-2}\end{aligned}$$

Equations for Correlation and Response

$$
\begin{aligned}
\frac{\partial^2 C(t,t')}{\partial t^2} &= -\mu_x(t)C(t,t') \\
&+ \frac{p_2}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_2-1} + \frac{p_2(p_2-1)}{2}\int_0^{t} du\, C(t',u)R(t,u)C(t,u)^{p_2-2} \\
&+ \beta_2\frac{p_2}{2}\Big(C(t',0)C(t,0)^{p_2-1} - \underline{K(0,t')K(0,t)^{p_2-1}}\Big) \\
&+ \frac{p_4}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_4-1} + \frac{p_4(p_4-1)}{2}\int_0^{t} du\, C(t',u)R(t,u)C(t,u)^{p_4-2} \\
&+ \beta_4\frac{p_4}{2}\Big(C(t',0)C(t,0)^{p_4-1} - \underline{K(0,t')K(0,t)^{p_4-1}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 R(t,t')}{\partial t^2} &= -\,\mu_x(t)R(t,t') + \frac{p_2(p_2-1)}{2}\int_{t'}^{t} du\, R(u,t')R(t,u)C(t,u)^{p_2-2} \\
&+ \frac{p_4(p_4-1)}{2}\int_{t'}^{t} du\, R(u,t')R(t,u)C(t,u)^{p_4-2}
\end{aligned}
$$

Where we introduced the **Pseudo-Correlation**

$$K(0,t) = \lim_{N\to\infty}\frac{1}{N}\sum_i \langle s_i(0)\rangle_{eq}\langle s_i(t)\rangle$$

And we **assumed**

self averaging of correlation, response and pseudo-correlation

mean-field approximation

Equations for Correlation and Response

$$\begin{aligned}\frac{\partial^2 C(t,t')}{\partial t^2} &= \underline{-\mu_x(t)}C(t,t') \\ &+ \frac{p_2}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_2-1} + \frac{p_2(p_2-1)}{2}\int_0^t du\, C(t',u)R(t,u)C(t,u)^{p_2-2} \\ &+ \beta_2\frac{p_2}{2}\Big(C(t',0)C(t,0)^{p_2-1} - \underline{K(0,t')K(0,t)^{p_2-1}}\Big) \\ &+ \frac{p_4}{2}\int_0^{t'} du R(t',u)C(t,u)^{p_4-1} + \frac{p_4(p_4-1)}{2}\int_0^t du\, C(t',u)R(t,u)C(t,u)^{p_4-2} \\ &+ \beta_4\frac{p_4}{2}\Big(C(t',0)C(t,0)^{p_4-1} - \underline{K(0,t')K(0,t)^{p_4-1}}\Big)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 R(t,t')}{\partial t^2} &= -\underline{\mu_x(t)}R(t,t') + \frac{p_2(p_2-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p_2-2} \\ &+ \frac{p_4(p_4-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p_4-2}\end{aligned}$$

Where we introduced the **Pseudo-Correlation**

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And we **assumed**

self averaging of correlation, response and pseudo-correlation

mean-field approximation

Equation for the Pseudo-Correlation

To determine the equation for the **Pseudo-Correlation**

$$K(0,t) = \lim_{N\to\infty} \frac{1}{N} \sum_i \langle s_i(0)\rangle_{eq} \langle s_i(t)\rangle$$

Main equation of dynamics:

$$\begin{aligned}
\mathbb{E}\langle \ddot{s}_i A(s(t'))\rangle = & -\mathbb{E}\langle \mu_x(t) s_i(t) A(s(t'))\rangle + \sum_j \mathbb{E}(J_{ij}^2)\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j A(s(t')) s_j(t)\rangle \\
& + \int_0^t du\, \mathbb{E}\langle s_i\, i\hat{s}_j A(s(t')) s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) A(s(t')) s_j(t)\rangle\Big] \\
& + \sum_{j<k<l} \mathbb{E}(J_{ijkl}^2)\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A(s(t')) s_j s_k s_l\rangle \\
& + \int_0^t du\, \mathbb{E}\langle s_i\, (i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A(s(t')) s_j s_k s_l\rangle \\
& + \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) A(s(t'))\, s_j s_k s_l\rangle\Big]
\end{aligned}$$

Equation for the Pseudo-Correlation

FPU problem

Potential Method

2+4 p-spin spherical Model

Potential Method

Disorder and replicas

Looking for minima

Hamiltonian dynamics

Generic equation of dynamics

Correlation and Response

Lagrangian multiplier

3=p-spin spherical Model

Potential Method

Correlation and Response

Future developments

To determine the equation for the **Pseudo-Correlation**

$$K(0,t) = \lim_{N\to\infty} \frac{1}{N} \sum_i \langle s_i(0)\rangle_{eq} \langle s_i(t)\rangle$$

We choose → $A(s_i(t')) = \langle s_i(0)\rangle_{eq}$

Main equation of dynamics:

$$\begin{aligned}
\mathbb{E}\langle \ddot{s}_i A\underline{(s(t'))}\rangle = &- \mathbb{E}\langle \mu_x(t) s_i(t) A\underline{(s(t'))}\rangle + \sum_j \mathbb{E}(J_{ij}^2)\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j A\underline{(s(t'))} s_j(t)\rangle \\
&+ \int_0^t du\, \mathbb{E}\langle s_i\, i\hat{s}_j A\underline{(s(t'))} s_j\rangle + \mathbb{E}\langle \beta_2 (s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) A\underline{(s(t'))} s_j(t)\rangle\Big] \\
&+ \sum_{j<k<l} \mathbb{E}(J_{ijkl}^2)\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A\underline{(s(t'))} s_j s_k s_l\rangle \\
&+ \int_0^t du\, \mathbb{E}\langle s_i\, (i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A\underline{(s(t'))} s_j s_k s_l\rangle \\
&+ \mathbb{E}\langle \beta_4 (s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) A\underline{(s(t'))}\, s_j s_k s_l\rangle\Big]
\end{aligned}$$

Equation for the Pseudo-Correlation

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&+ \int_0^t du\, \mathbb{E}\langle s_i\, i\hat{s}_j A\underline{(s(t'))} s_j\rangle + \mathbb{E}\langle \beta_2(s_i^0 s_j^0 + \langle s_i^0 s_j^0\rangle_{eq}) \underline{A(s(t'))} s_j(t)\rangle\Big] \\
&+ \sum_{j<k<l}\mathbb{E}(J_{ijkl}^2)\Big[\int_0^t du\, \mathbb{E}\langle i\hat{s}_i(u) s_j s_k s_l A\underline{(s(t'))} s_j s_k s_l\rangle \\
&+ \int_0^t du\, \mathbb{E}\langle s_i\,(i\hat{s}_j s_k s_l + s_j\, i\hat{s}_k s_l + s_j s_k\, i\hat{s}_l) A\underline{(s(t'))} s_j s_k s_l\rangle \\
&+ \mathbb{E}\langle \beta_4(s_i^0 s_j^0 s_k^0 s_l^0 + \langle s_i^0 s_j^0 s_k^0 s_l^0\rangle_{eq}) \underline{A(s(t'))}\, s_j s_k s_l\rangle\Big]
\end{aligned}$$

The differential equation for the **Pseudo-Correlation**

$$\boxed{\begin{aligned}
\frac{\partial^2 K(0,t)}{\partial t^2} = &-\mu_x(t)K(0,t) + \frac{p_2(p_2-1)}{2}\int_0^t du\, K(0,u)R(t,u)C(t,u)^{p_2-2} + \beta_2\frac{p_2}{2}\Big(K(0,0)C(t,0)^{p_2-1} - \bar{q}K(0,t)^{p_2-1}\Big) \\
&+ \frac{p_4(p_4-1)}{2}\int_0^t du\, K(0,u)R(t,u)C(t,u)^{p_4-2} + \beta_4\frac{p_4}{2}\Big(K(0,0)C(t,0)^{p_4-1} - \bar{q}K(0,t)^{p_4-1}\Big)
\end{aligned}}$$

Equation for the lagrangian multiplier

Hamiltonian:

$$H(s) = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N) \quad \longrightarrow \quad \sum_i p_i^2 = 2(H - V_J)$$

Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{s}_i = p_i$$

$$\frac{\partial H}{\partial s_i} = -\dot{p}_i = \frac{\partial V}{\partial s_i} + \mu_x s_i$$

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$$\sum_i \dot{p}_i s_i = \sum_i \ddot{s}_i s_i = \frac{1}{2}\frac{d}{dt}\sum_i s_i^2 - \sum_i \dot{s}_i^2 = -\sum_i \dot{s}_i^2$$

Equation for the lagrangian multiplier

Hamiltonian:

$$H(s) = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N) \quad \longrightarrow \quad \sum_i p_i^2 = 2(H - V_J)$$

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$$\sum_i \dot{p}_i s_i = \sum_i \ddot{s}_i s_i = \frac{1}{2}\frac{d}{dt}\underbrace{\sum_i s_i^2}_{N} - \sum_i \dot{s}_i^2 = -\sum_i \dot{s}_i^2$$

Equation for the lagrangian multiplier

Hamiltonian:

$$H(s) = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N) \quad \longrightarrow \quad \sum_i p_i^2 = 2(H - V_J)$$

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$$\sum_i \dot{p}_i s_i = \sum_i \ddot{s}_i s_i = \frac{1}{2}\frac{d}{dt}\underbrace{\sum_i s_i^2}_{N} - \sum_i \dot{s}_i^2 = -\sum_i \dot{s}_i^2$$

$$\sum_i \ddot{s}_i s_i = -\sum_i \frac{\partial V}{\partial s_i} s_i - N\mu_x = -\sum_i \dot{s}_i^2 = \sum_i p_i^2 = 2(H - V_J)$$

Equation for the lagrangian multiplier

Hamiltonian:

$$H(s) = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \frac{\mu_x(t)}{2}(\sum_i s_i^2 - N) \rightarrow \sum_i p_i^2 = 2(H - V_J)$$

Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{s}_i = p_i$$

$$\frac{\partial H}{\partial s_i} = -\dot{p}_i = \frac{\partial V}{\partial s_i} + \mu_x s_i$$

$$\sum_i \dot{p}_i s_i = \sum_i \ddot{s}_i s_i = \frac{1}{2}\frac{d}{dt}\underset{N}{\boxed{\sum_i s_i^2}} - \sum_i \dot{s}_i^2 = -\sum_i \dot{s}_i^2$$

$$\sum_i \ddot{s}_i s_i = -\underline{\sum_i \frac{\partial V}{\partial s_i} s_i - N\mu_x} = -\sum_i \dot{s}_i^2 = \sum_i p_i^2 = \underline{2(H - V_J)}$$

Equation for the lagrangian multiplier

Hamiltonian:

$$H(s) = \frac{1}{2}\sum_i p_i^2 + V_J(s) + \frac{\mu_x(t)}{2}\Big(\sum_i s_i^2 - N\Big) \quad \longrightarrow \quad \textstyle\sum_i p_i^2 = 2(H - V_J)$$

Hamilton's equations

$$\frac{\partial H}{\partial p_i} = \dot{s}_i = p_i$$

$$\frac{\partial H}{\partial s_i} = -\dot{p}_i = \frac{\partial V}{\partial s_i} + \mu_x s_i$$

$$\sum_i \dot{p}_i s_i = \sum_i \ddot{s}_i s_i = \frac{1}{2}\frac{d}{dt}\underbrace{\sum_i s_i^2}_{N} - \sum_i \dot{s}_i^2 = -\sum_i \dot{s}_i^2$$

$$\sum_i \ddot{s}_i s_i = -\underline{\sum_i \frac{\partial V}{\partial s_i} s_i - N\mu_x} = -\sum_i \dot{s}_i^2 = \sum_i p_i^2 = \underline{2(H - V_J)}$$

We obtain the relation desired

$$N\mu_x = -\sum_i \frac{\partial V_J}{\partial s_i} s_i + 2(H - V_J)$$

Averaging

$$N\mu_x = -\sum_i \mathbb{E}\Big\langle \frac{\partial V_J}{\partial s_i} s_i \Big\rangle + 2(E - \mathbb{E}\langle V_J \rangle)$$

Equation for the lagrangian multiplier

$$\boxed{N\mu_x = -\sum_i \mathbb{E}\Big\langle \frac{\partial V_J}{\partial s_i} s_i \Big\rangle + 2(E - \mathbb{E}\langle V_J \rangle)}$$

Equation for the lagrangian multiplier

$$\boxed{N\mu_x = -\sum_i \mathbb{E}\Big\langle \frac{\partial V_J}{\partial s_i} s_i \Big\rangle + 2(E - \mathbb{E}\langle V_J \rangle)}$$

With computations analogous to those previously seen for correlation and response

Equation for the **lagrangian multiplier** that enforces the **spherical constraint**

$$\boxed{\begin{aligned}\mu_x(t) =& 2e + 4\frac{p_2}{2}\int_0^t du R(t,u)C(t,u)^{p_2-1} + \beta_2\Big(C(t,0)^{p_2} - K(0,t)^{p_2}\Big)\\ &+ 6\frac{p_4}{2}\int_0^t du R(t,u)C(t,u)^{p_4-1} + \beta_4\Big(C(t,0)^{p_4} - K(0,t)^{p_4}\Big)\end{aligned}}$$

Equations for dynamics

Correlation

$$\begin{aligned}\frac{\partial^2 C(t,t')}{\partial t^2} &= -\mu_x(t)C(t,t') \\ &\quad + \frac{p_2}{2}\int_0^{t'} duR(t',u)C(t,u)^{p_2-1} + \frac{p_2(p_2-1)}{2}\int_0^{t} du\, C(t',u)R(t,u)C(t,u)^{p_2-2} \\ &\quad + \beta_2\frac{p_2}{2}\Big(C(t',0)C(t,0)^{p_2-1} - K(0,t')K(0,t)^{p_2-1}\Big) \\ &\quad + \frac{p_4}{2}\int_0^{t'} duR(t',u)C(t,u)^{p_4-1} + \frac{p_4(p_4-1)}{2}\int_0^{t} du\, C(t',u)R(t,u)C(t,u)^{p_4-2} \\ &\quad + \beta_4\frac{p_4}{2}\Big(C(t',0)C(t,0)^{p_4-1} - K(0,t')K(0,t)^{p_4-1}\Big)\end{aligned}$$

Response

$$\begin{aligned}\frac{\partial^2 R(t,t')}{\partial t^2} &= -\,\mu_x(t)R(t,t') + \frac{p_2(p_2-1)}{2}\int_{t'}^{t} du\, R(u,t')R(t,u)C(t,u)^{p_2-2} \\ &\quad + \frac{p_4(p_4-1)}{2}\int_{t'}^{t} du\, R(u,t')R(t,u)C(t,u)^{p_4-2}\end{aligned}$$

Pseudo-Correlation

$$\begin{aligned}\frac{\partial^2 K(0,t)}{\partial t^2} &= -\,\mu_x(t)K(0,t) + \frac{p_2(p_2-1)}{2}\int_0^{t} du\, K(0,u)R(t,u)C(t,u)^{p_2-2} + \beta_2\frac{p_2}{2}\Big(K(0,0)C(t,0)^{p_2-1} - \bar{q}K(0,t)^{p_2-1}\Big) \\ &\quad + \frac{p_4(p_4-1)}{2}\int_0^{t} du\, K(0,u)R(t,u)C(t,u)^{p_4-2} + \beta_4\frac{p_4}{2}\Big(K(0,0)C(t,0)^{p_4-1} - \bar{q}K(0,t)^{p_4-1}\Big)\end{aligned}$$

Lagrangian multiplier

$$\begin{aligned}\mu_x(t) =& 2e + 4\frac{p_2}{2}\int_0^{t} duR(t,u)C(t,u)^{p_2-1} + \beta_2\Big(C(t,0)^{p_2} - K(0,t)^{p_2}\Big) \\ &+ 6\frac{p_4}{2}\int_0^{t} duR(t,u)C(t,u)^{p_4-1} + \beta_4\Big(C(t,0)^{p_4} - K(0,t)^{p_4}\Big)\end{aligned}$$

P=3 spin spherical model partitioned in two subsystems

FPU problem
Potential Method
2+4 p-spin spherical Model
Potential Method
Disorder and replicas
Looking for minima
Hamiltonian dynamics
Generic equation of dynamics
Correlation and Response
Lagrangian multiplier
3=p-spin spherical Model
Potential Method
Correlation and Response
Future developments

$$
\begin{aligned}
V_J =& H_1 + H_{12} + H_{21} + H_2 \\
=& - \sum_{i<j<k}^{\gamma N, \gamma N, \gamma N} J_{ijk}^{(1)}\, s_i^{(1)} s_j^{(1)} s_k^{(1)} - \sum_{i<j,k}^{\substack{\gamma N, \gamma N, \\ (1-\gamma)N}} J_{ijk}^{(12)}\, s_i^{(1)} s_j^{(1)} s_k^{(2)} \quad - \sum_{i<j,k}^{\substack{\gamma N, (1-\gamma)N, \\ (1-\gamma)N}} J_{ijk}^{(21)}\, s_i^{(1)} s_j^{(2)} s_k^{(2)} - \sum_{i<j<k}^{\substack{(1-\gamma)N, (1-\gamma)N, \\ (1-\gamma)N}} J_{ijk}^{(2)}\, s_i^{(2)} s_j^{(2)} s_k^{(2)},
\end{aligned}
$$

P=3 spin spherical model partitioned in two subsystems

$$\begin{aligned} V_J =& H_1 + H_{12} + H_{21} + H_2 \\ =& -\sum_{i<j<k}^{\gamma N,\gamma N,\gamma N} J^{(1)}_{ijk}\, s_i^{(1)} s_j^{(1)} s_k^{(1)} - \sum_{i<j,k}^{\substack{\gamma N,\gamma N,\\ (1-\gamma)N}} J^{(12)}_{ijk}\, s_i^{(1)} s_j^{(1)} s_k^{(2)} - \sum_{i<j,k}^{\substack{\gamma N,(1-\gamma)N,\\ (1-\gamma)N}} J^{(21)}_{ijk}\, s_i^{(1)} s_j^{(2)} s_k^{(2)} - \sum_{i<j<k}^{\substack{(1-\gamma)N,(1-\gamma)N,\\ (1-\gamma)N}} J^{(2)}_{ijk}\, s_i^{(2)} s_j^{(2)} s_k^{(2)}, \end{aligned}$$

Replicated partition function

$$Z^{(n,m)} = \overline{\int Ds^a \int D\sigma^\alpha \exp\Big[\beta' \sum_a^n H(s^a) + \beta \sum_\alpha^m H(\sigma^\alpha)\Big] \prod_{\alpha=1}^m \delta\big(\sum_i^{\gamma N} s_i^{1(1)} \sigma_i^{\alpha(1)} - N\tilde{p}_1\big)\delta\big(\sum_i^{(1-\gamma)N} s_i^{1(2)} \sigma_i^{\alpha(2)} - N\tilde{p}_2\big)}$$

P=3 spin spherical model partitioned in two subsystems

$$\begin{aligned} V_J =& H_1 + H_{12} + H_{21} + H_2 \\ =& -\sum_{i<j<k}^{\gamma N,\gamma N,\gamma N} J_{ijk}^{(1)}\, s_i^{(1)} s_j^{(1)} s_k^{(1)} - \sum_{i<j,k}^{\substack{\gamma N,\gamma N,\\(1-\gamma)N}} J_{ijk}^{(12)}\, s_i^{(1)} s_j^{(1)} s_k^{(2)} - \sum_{i<j,k}^{\substack{\gamma N,(1-\gamma)N,\\(1-\gamma)N}} J_{ijk}^{(21)}\, s_i^{(1)} s_j^{(2)} s_k^{(2)} - \sum_{i<j<k}^{\substack{(1-\gamma)N,(1-\gamma)N,\\(1-\gamma)N}} J_{ijk}^{(2)}\, s_i^{(2)} s_j^{(2)} s_k^{(2)}\,, \end{aligned}$$

Replicated partition function

$$Z^{(n,m)} = \overline{\int Ds^a \int D\sigma^\alpha \exp\left[\beta' \sum_a^n H(s^a) + \beta \sum_\alpha^m H(\sigma^\alpha)\right] \prod_{\alpha=1}^m \underbrace{\delta\Big(\sum_i^{\gamma N} s_i^{1(1)} \sigma_i^{\alpha(1)} - N\tilde{p}_1\Big)}_{\text{fixed distance between the two subsystems 1}} \underbrace{\delta\Big(\sum_i^{(1-\gamma)N} s_i^{1(2)} \sigma_i^{\alpha(2)} - N\tilde{p}_2\Big)}_{\text{fixed distance between the two subsystems 2}}}$$

P=3 spin spherical model partitioned in two subsystems

$$\begin{aligned} V_J =& H_1 + H_{12} + H_{21} + H_2 \\ =& -\sum_{i<j<k}^{\gamma N,\gamma N,\gamma N} J^{(1)}_{ijk}\, s^{(1)}_i s^{(1)}_j s^{(1)}_k - \sum_{i<j,k}^{\substack{\gamma N,\gamma N,\\(1-\gamma)N}} J^{(12)}_{ijk}\, s^{(1)}_i s^{(1)}_j s^{(2)}_k - \sum_{i<j,k}^{\substack{\gamma N,(1-\gamma)N,\\(1-\gamma)N}} J^{(21)}_{ijk}\, s^{(1)}_i s^{(2)}_j s^{(2)}_k - \sum_{i<j<k}^{\substack{(1-\gamma)N,(1-\gamma)N,\\(1-\gamma)N}} J^{(2)}_{ijk}\, s^{(2)}_i s^{(2)}_j s^{(2)}_k \,, \end{aligned}$$

Replicated partition function

$$Z^{(n,m)} = \overline{\int Ds^a \int D\sigma^\alpha \exp\left[\beta' \sum_a^n H(s^a) + \beta \sum_\alpha^m H(\sigma^\alpha)\right] \prod_{\alpha=1}^m \underbrace{\delta\Big(\sum_i^{\gamma N} s_i^{1(1)}\sigma_i^{\alpha(1)} - N\tilde{p}_1\Big)}_{\text{fixed distance between the two subsystems 1}} \underbrace{\delta\Big(\sum_i^{(1-\gamma)N} s_i^{1(2)}\sigma_i^{\alpha(2)} - N\tilde{p}_2\Big)}_{\text{fixed distance between the two subsystems 2}}}$$

- Average over disorder

P=3 spin spherical model partitioned in two subsystems

$$\begin{aligned} V_J =& H_1 + H_{12} + H_{21} + H_2 \\ =& -\sum_{i<j<k}^{\gamma N,\gamma N,\gamma N} J_{ijk}^{(1)}\, s_i^{(1)} s_j^{(1)} s_k^{(1)} - \sum_{i<j,k}^{\substack{\gamma N,\gamma N,\\(1-\gamma)N}} J_{ijk}^{(12)}\, s_i^{(1)} s_j^{(1)} s_k^{(2)} - \sum_{i<j,k}^{\substack{\gamma N,(1-\gamma)N,\\(1-\gamma)N}} J_{ijk}^{(21)}\, s_i^{(1)} s_j^{(2)} s_k^{(2)} - \sum_{i<j<k}^{\substack{(1-\gamma)N,(1-\gamma)N,\\(1-\gamma)N}} J_{ijk}^{(2)}\, s_i^{(2)} s_j^{(2)} s_k^{(2)}, \end{aligned}$$

Replicated partition function

$$Z^{(n,m)} = \overline{\int Ds^a \int D\sigma^\alpha \exp\left[\beta' \sum_a^n H(s^a) + \beta \sum_\alpha^m H(\sigma^\alpha)\right] \prod_{\alpha=1}^m \underbrace{\delta\Big(\sum_i^{\gamma N} s_i^{1(1)} \sigma_i^{\alpha(1)} - N\tilde{p}_1\Big)}_{\substack{\text{fixed distance between}\\ \text{the two subsystems 1}}} \underbrace{\delta\Big(\sum_i^{(1-\gamma)N} s_i^{1(2)} \sigma_i^{\alpha(2)} - N\tilde{p}_2\Big)}_{\substack{\text{fixed distance between}\\ \text{the two subsystems 2}}}}$$

- Average over disorder

Introduce

Order parameter matrices

$$Q_{ab}^{(1)} = \frac{1}{\gamma N} \sum_i s_i^{(1)a} s_i^{(1)b} \qquad Q_{ab}^{(2)} = \frac{1}{(1-\gamma)N} \sum_i s_i^{(2)a} s_i^{(2)b}$$

$$R_{\alpha\beta}^{(1)} = \frac{1}{\gamma N} \sum_i \sigma_i^{(1)\alpha} \sigma_i^{(1)\beta} \qquad R_{\alpha\beta}^{(2)} = \frac{1}{(1-\gamma)N} \sum_i \sigma_i^{(2)\alpha} \sigma_i^{(2)\beta}$$

$$P_{a\alpha}^{(1)} = \frac{1}{\gamma N} \sum_i s_i^{(1)a} \sigma_i^{(1)\alpha} \qquad P_{a\alpha}^{(2)} = \frac{1}{(1-\gamma)N} \sum_i s_i^{(2)a} \sigma_i^{(2)\alpha}$$

Single matrices for system 1 and 2

$$\mathbf{Q}^{(1)} = \begin{pmatrix} Q^{(1)} & P^{(1)} \\ P^{(1)\,T} & R^{(1)} \end{pmatrix}$$

$$\mathbf{Q}^{(2)} = \begin{pmatrix} Q^{(2)} & P^{(2)} \\ P^{(2)\,T} & R^{(2)} \end{pmatrix}$$

Generalized RS ansatz

Using a **saddle point** technique to estimate the integral

$$Z^{(n,m)} = \int D\mathbf{Q}_{\gamma\eta} \int D\lambda_{\gamma\eta} \exp[-NS(\lambda, \mathbf{Q})] \simeq \exp[-NS(\lambda^*, \mathbf{Q}^*)]$$

$$\boxed{\begin{aligned} \frac{1}{N}\ln Z^{n,m} = &+ \frac{1}{4}\gamma^3\Big(\beta_1^2 \sum_{a,b}^{n} Q_{ab}^{(1)3} + 2\beta_1\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)3} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)3}\Big) \ + \frac{1}{4}(1-\gamma)^3\Big(\beta_2^2 \sum_{a,b}^{n} Q_{ab}^{(2)3} + 2\beta_2\beta \sum_{a,\alpha} P_{a,\alpha}^{(2)3} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(2)3}\Big) \\ &+ \frac{3}{4}\gamma^2(1-\gamma)\Big(\beta_{12}^2 \sum_{a,b}^{n} Q_{ab}^{(1)2} Q_{ab}^{(2)} + 2\beta_{12}\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)2} P_{a,\alpha}^{(2)} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)2} R_{\alpha,\beta}^{(2)}\Big) \\ &+ \frac{3}{4}\gamma(1-\gamma)^2\Big(\beta_{21}^2 \sum_{a,b}^{n} Q_{ab}^{(1)} Q_{ab}^{(2)2} + 2\beta_{21}\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)} P_{a,\alpha}^{(2)2} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)} R_{\alpha,\beta}^{(2)2}\Big) \\ &+ \frac{1}{2}\ln\det\begin{pmatrix} Q^{(1)} & P^{(1)} \\ P^{(1)\,T} & R^{(1)} \end{pmatrix} + \frac{1}{2}\ln\det\begin{pmatrix} Q^{(2)} & P^{(2)} \\ P^{(2)\,T} & R^{(2)} \end{pmatrix} \end{aligned}}$$

The Effective Potential can be obtained using

$$NV = -T\frac{\partial}{\partial m}\ln Z^{(n,m)}\Big|_{\substack{m=0\\n=0}}$$

Generalized RS ansatz

Using a **saddle point** technique to estimate the integral

$$Z^{(n,m)} = \int D\mathbf{Q}_{\gamma\eta} \int D\lambda_{\gamma\eta} \exp[-NS(\lambda, \mathbf{Q})] \simeq \exp[-NS(\lambda^*, \mathbf{Q}^*)]$$

$$\boxed{\begin{aligned} \frac{1}{N}\ln Z^{n,m} = &+ \frac{1}{4}\gamma^3\Big(\beta_1^2 \sum_{a,b}^{n} Q_{ab}^{(1)3} + 2\beta_1\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)3} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)3}\Big) \; + \frac{1}{4}(1-\gamma)^3\Big(\beta_2^2 \sum_{a,b}^{n} Q_{ab}^{(2)3} + 2\beta_2\beta \sum_{a,\alpha} P_{a,\alpha}^{(2)3} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(2)3}\Big) \\ &+ \frac{3}{4}\gamma^2(1-\gamma)\Big(\beta_{12}^2 \sum_{a,b}^{n} Q_{ab}^{(1)2} Q_{ab}^{(2)} + 2\beta_{12}\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)2} P_{a,\alpha}^{(2)} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)2} R_{\alpha,\beta}^{(2)}\Big) \\ &+ \frac{3}{4}\gamma(1-\gamma)^2\Big(\beta_{21}^2 \sum_{a,b}^{n} Q_{ab}^{(1)} Q_{ab}^{(2)2} + 2\beta_{21}\beta \sum_{a,\alpha} P_{a,\alpha}^{(1)} P_{a,\alpha}^{(2)2} + \beta^2 \sum_{\alpha,\beta} R_{\alpha,\beta}^{(1)} R_{\alpha,\beta}^{(2)2}\Big) \\ &+ \frac{1}{2}\ln\det\begin{pmatrix} Q^{(1)} & P^{(1)} \\ P^{(1)\,T} & R^{(1)} \end{pmatrix} + \frac{1}{2}\ln\det\begin{pmatrix} Q^{(2)} & P^{(2)} \\ P^{(2)\,T} & R^{(2)} \end{pmatrix} \end{aligned}}$$

The Effective Potential can be obtained using

$$NV = -T\frac{\partial}{\partial m}\ln Z^{(n,m)}\Big|_{\substack{m=0\\n=0}}$$

RS Ansatz

for the Overlap Matrices

$\longrightarrow$

$$\mathbf{Q} = \begin{pmatrix} Q & P \\ P^T & R \end{pmatrix} = \begin{pmatrix} \overbrace{\begin{matrix} 1 & q & \cdots & q \\ q & 1 & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & 1 \end{matrix}}^{n} & \overbrace{\begin{matrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \tilde{p} & \tilde{p} & \cdots & \tilde{p} \end{matrix}}^{m} \\ \begin{matrix} 0 & \cdots & 0 & \tilde{p} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \tilde{p} \\ 0 & \cdots & 0 & \tilde{p} \end{matrix} & \begin{matrix} 1 & r & \cdots & r \\ r & 1 & \cdots & r \\ \vdots & \vdots & \ddots & \vdots \\ r & r & \cdots & 1 \end{matrix} \end{pmatrix}$$

Effective Potential

Potential Function

$$\begin{aligned}\beta V(p_1, r_1, p_2, r_2) = &-\frac{1}{4}\gamma^3(\beta^2 + 2\beta_1\beta\, p_1^3 - \beta^2\, r_1^3)\\ &-\frac{1}{4}(1-\gamma)^3(\beta^2 + 2\beta_2\beta\, p_2^3 - \beta^2\, r_2^3)\\ &-\frac{3}{4}(1-\gamma)^2\gamma(\beta^2 + 2\beta_{21}\beta\, p_1\, p_2^2 - \beta^2 r_1\, r_2^2)\\ &-\frac{3}{4}\gamma^2(1-\gamma)(\beta^2 + 2\beta_{12}\beta\, p_1^2\, p_2 - \beta^2 r_1^2\, r_2)\\ &-\frac{1}{2}\gamma\left(\frac{r_1 - p_1^2}{1 - r1} + \log[1 - r_1]\right) - \frac{1}{2}(1-\gamma)\left(\frac{r_2 - p_2^2}{1 - r_2} + \log[1 - r_2]\right)\end{aligned}$$

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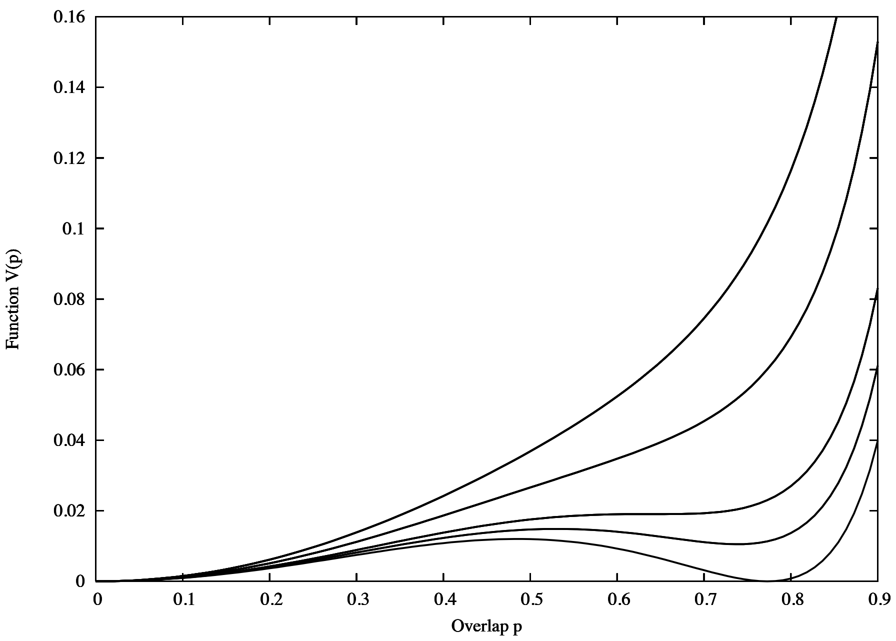

Hamiltonian dynamics

Equation for the Hamiltonian dynamics of a generic p-spin spherical model

Correlation

$$\begin{aligned}\frac{\partial^2 C(t,t')}{\partial t^2} &= -\mu(t)C(t,t') \\ &+ \frac{p}{2}\int_0^{t'} duR(t',u)C(t,u)^{p-1} + \frac{p(p-1)}{2}\int_0^t du\, C(t',u)R(t,u)C(t,u)^{p-2} \\ &+ \beta'\frac{p}{2}\Big(C(t',0)C(t,0)^{p-1} - K(0,t')K(0,t)^{p-1}\Big)\end{aligned}$$

Response

$$\begin{aligned}\frac{\partial^2 R(t,t')}{\partial t^2} &= -\,\mu(t)R(t,t') + \frac{p(p-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p-2} \\ &+ \frac{p(p-1)}{2}\int_{t'}^t du\, R(u,t')R(t,u)C(t,u)^{p-2}\end{aligned}$$

Pseudo-Correlation

$$\begin{aligned}\frac{\partial^2 K(0,t)}{\partial t^2} &= -\,\mu_x(t)K(0,t) + \frac{p(p-1)}{2}\int_0^t du\, K(0,u)R(t,u)C(t,u)^{p-2} \\ &+ \beta'\frac{p}{2}\Big(K(0,0)C(t,0)^{p-1} - \bar{q}K(0,t)^{p-1}\Big)\end{aligned}$$

Substituting:

$$\begin{gathered}C(t,t') \to \gamma C_1(t,t') + (1-\gamma)C_2(t,t') \\ R(t,t') \to \gamma R_1(t,t') + (1-\gamma)R_2(t,t') \\ K(t,t') \to \gamma K_1(t,t') + (1-\gamma)K_2(t,t') \\ \bar{q} \to \gamma\bar{q}_1 + (1-\gamma)\bar{q}_2\end{gathered}$$

Equations for the dynamics

FPU problem

Potential Method

2+4 p-spin spherical Model

Potential Method

Disorder and replicas

Looking for minima

Hamiltonian dynamics

Generic equation of dynamics

Correlation and Response

Lagrangian multiplier

3=p-spin spherical Model

Potential Method

Correlation and Response

Future developments

Correlation

$$
\begin{aligned}
\frac{\partial^2 C_1(t,t')}{\partial t^2} = &-\gamma\mu C_1(t,t') \\
&+\frac{p(p-1)}{2}\left\{\gamma^3\int_0^t du C_1(t,u)R_1(t,u)\Big(C_1(t,u)+C_1(t',u)\Big) + \gamma^2(1-\gamma)\int_0^t du C_1(t',u)\Big(C_2(t,u)R_1(t,u)+C_1(t,u)R_2(t,u)\Big)\right. \\
&\left.\qquad + \gamma^2(1-\gamma)\int_0^t du C_1(t,u)\Big(C_2(t',u)R_1(t,u)+C_2(t,u)R_1(t',u)+C_1(t,u)R_2(t',u)\Big)\right\} \\
&+\frac{p}{2}\Big\{\beta_1\gamma^3\Big(C_1(t,0)^2C_1(t',0)-K_1(t,0)^2K_1(t',0)\Big) + 2\beta_{12}\gamma^2(1-\gamma)\Big(C_1(t,0)C_1(t',0)C_2(t,0)-K_1(t,0)K_1(t',0)K_2(t,0)\Big) \\
&\qquad + \beta_{12}\gamma^2(1-\gamma)\Big(C_1(t,0)^2C_2(t',0)-K_1(t,0)^2K_2(t',0)\Big)\Big\}
\end{aligned}
$$

Response

$$
\begin{aligned}
\frac{\partial^2 R_1(t,t')}{\partial t^2} = &-\mu(t)\gamma R_1(t,t') + \frac{p(p-1)}{2}\left[\int_{t'}^t du\, C_1(t,u)R_1(t,u)\Big(\gamma^3 R_1(u,t')+\gamma^2(1-\gamma)R_2(u,t')\Big)\right. \\
&\left.\qquad + \gamma^2(1-\gamma)\int_{t'}^t du\, R_1(u,t')\Big(C_2(t,u)R_1(t,u)+C_1(t,u)R_2(t,u)\Big)\right]
\end{aligned}
$$

Pseudo-Correlation

$$
\begin{aligned}
\frac{\partial^2 K_1(t,t')}{\partial t^2} = &-\gamma\mu(t)K_1(0,t) \\
&+\frac{p(p-1)}{2}\left[\int_0^t du\, K_1(0,u)R_1(t,u)\Big(\gamma^3C_1(t,u)+\gamma^2(1-\gamma)C_2(t,u)\Big) + \gamma^2(1-\gamma)\int_0^t \mathrm{du}C_1(t,u)\Big(K_2(0,u)R_1(t,u)+K_1(0,u)R_2(t,u)\Big)\right] \\
&+\frac{p}{2}\Big\{\beta_1\gamma^3\big(C_1(t,0)^2K_1(0,0)-q_1K_1(0,t)^2\big) + \beta_{12}\gamma^2(1-\gamma)\Big[\big(C_1(t,0)^2K_2(0,0)-q_2K_1(0,t)^2\big) \\
&\qquad + 2\left(C_1(t,0)C_2(t,0)K_1(0,0)-q_1K_1(0,t)K_2(0,t)\right)\Big]\Big\}
\end{aligned}
$$

Further developments

- 1-RSB treatment of the potential for the static formulation in the 2+3 spin spherical model

- Full RSB treatment of the potential for the static formulation in the 2+4 spin spherical model

- Numerical results for the integro-differential equations for correlation, response, pseudo-correlation and lagrangian multiplier

- Comparison between static (using the effective potential) and dynamic results

Hopefully find some connections between
Ergodicity Breaking in FPU and Spin Glasses

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