Systemic Risk and the Mathematics of Falling Dominoes

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NETADIS
Statistical Physics Approaches to Networks Across Disciplines

Outline

The Laws of Falling Dominoes

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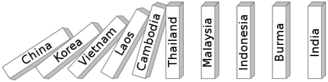
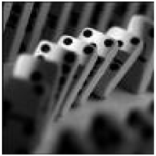
- Avalanches can occur, if dominoes are set too closely.

Risk and Falling Dominoes

Operational Risk



Domino Theory & Spread of Communism



Blackouts in Power Grids

Financial Crisis

Fundamental Problem of Risk Analysis

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- rare fluctuations in cash-flows, requiring short term acquisition of funds to maintain liquidity $\leftrightarrow$ liquidity risk

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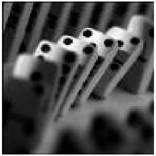

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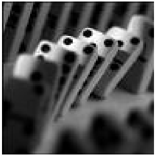

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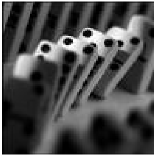
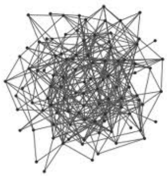
  - Volatility clustering in markets (intermittency)



Operational Risks — Interacting Processes

- Conceptualise organisation as a network of processes
- Two state model: processes either up and running ($n_i = 0$) or down ($n_i = 1$)
- Reliability of processes and degree of functional interdependence heterogeneous across the set of processes; connectivity & concept of neighbourhood functionally defined

$\Rightarrow$model defined on random graph



- losses determined (randomly) each time a process goes down

Dynamics – Mathematics of Falling Dominoes

- Processes need support to keep running (energy, human resources, material, information, input from other processes, etc.)
- h_{it} total support received by process i at time t

$$h_{it} = h_i^* - \sum_j J_{ij} n_{jt} + x_{it}$$

  - h_i^* support in fully functional environment
  - J_{ij} support to process i provided by process j
  - x_{it} random (e.g. Gaussian white noise).

- Process i will fail, if the total support for it falls below a critical threshold (if $h_{it} \leq 0$ – domino falls, if kicked too strongly)

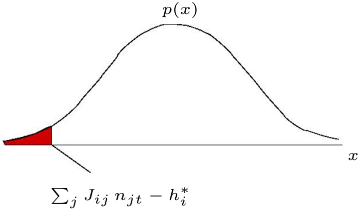
$$n_{it+1} = \Theta\big(-h_{it}\big) = \Theta\Big(\sum_j J_{ij} n_{jt} - h_i^* - x_{it}\Big)$$

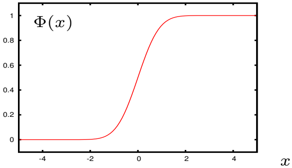
Because of the random noise x_{it}, failure is a probabilistic event.

Probability that a Domino Falls

- Probability of failure/probability of domino falling

$$\text{Prob}\big(n_{it+1} = 1\big) = \text{Prob}\big(x_{it} < \textstyle\sum_j J_{ij}\, n_{jt} - h_i^*\big) \equiv \Phi\big(\textstyle\sum_j J_{ij}\, n_{jt} - h_i^*\big)$$





- unconditional and conditional probability of failure

$$p_i = \Phi(-h_i^*) \quad , \qquad p_{i|j} = \Phi\Big(J_{ij} - h_i^*\Big)$$

A Simple Homogeneous Process Network

- Large homogeneous system $1 \le i \le N$.

  Uniform all-to-all couplings $J_{ij} = J_0/N$, and $h_i^* = h^*$ indep. of i.

$$\Rightarrow \quad \sum_j J_{ij} n_{jt} = \frac{J_0}{N} \sum_j n_{jt} = J_0 m_t$$

- Dynamics depends only on fraction m_t of failed nodes.

$$n_{it+1} = \Theta\Big(\sum_j J_{ij} n_{jt} - h_i^* - x_{it}\Big) = \Theta\Big(J_0 m_t - h^* - x_{it}\Big) \ .$$

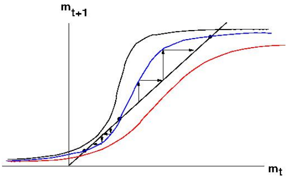
  Thus by Law of Large Numbers (LLN)

$$m_{t+1} = \frac{1}{N} \sum_{i=1}^{N} \Theta\Big(J_0 m_t - h^* - x_{it}\Big) \simeq \Phi\Big(J_0 m_t - h^*\Big)$$

Analysis of the Dynamics

- Iterated function dynamics

$$m_{t+1} = \Phi\Big(J_0 m_t - h^*\Big)$$



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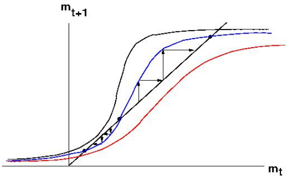

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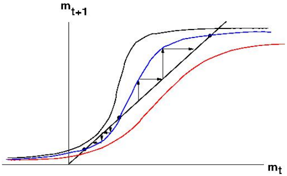

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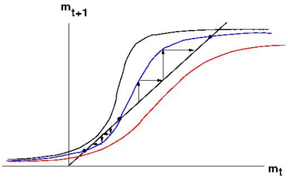

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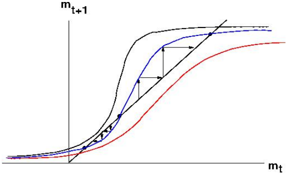

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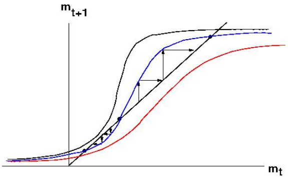

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- $\Phi\Big(J_0 m_t - h^*\Big)$ as a function of m_t has inflection point at $m_t = h^*/J_0$ and (maximum) slope

$$J_0 \Phi'(0) \quad , \quad (> 1 \text{for sufficiently large } J_0)$$

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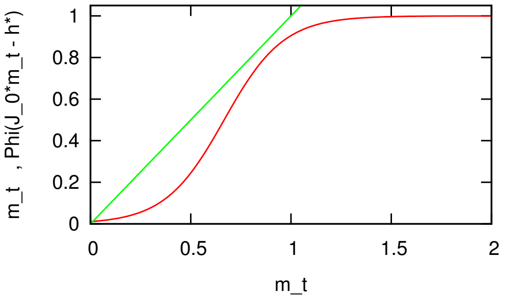

Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and $J_0 = 3$

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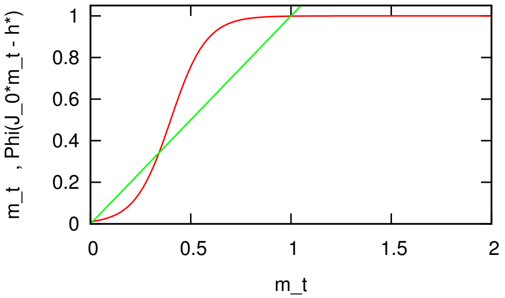

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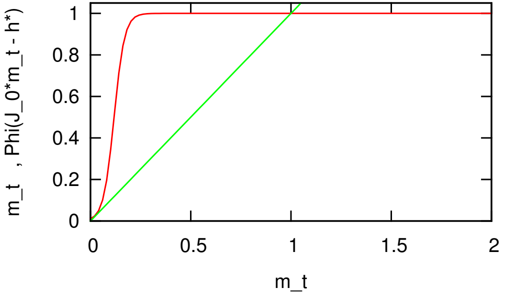

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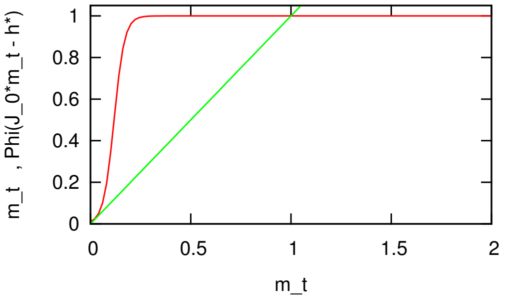
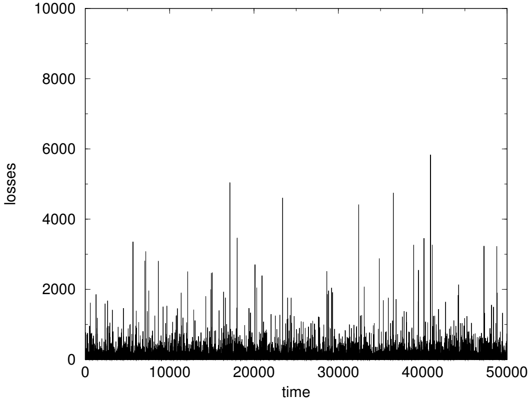


Graphical anlysis of stationary solution $m = \Phi(J_0 m - h^*)$ for $h^* = 2$ and

- For not too small values of h^* can change from system with only low-m, via system with coexisting low-m and high-m states, to system with only high-m states by increasing J_0. For small h^* have only high-m state.

Spontaneous Breakdown



Losses from operational risks in a network of 100 processes: J_0 such that low-m solution is stable

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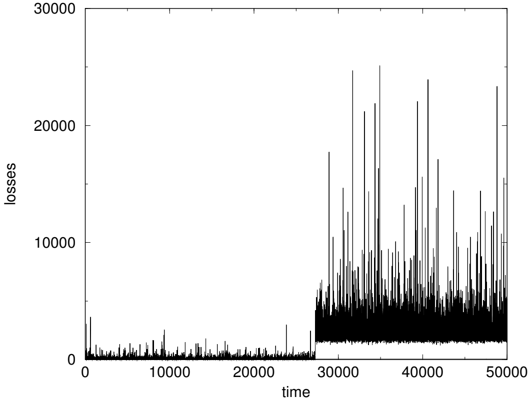

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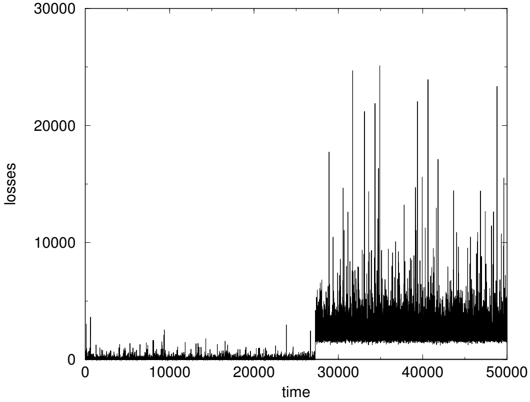

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- Spontaneous breakdown of meta-stable functioning solution possible in finite systems

Summary

- We found that networks can be destabilized by large degrees of interdependency (large J_0) even if all processes are very reliable (with large h^*).
- For intermediate levels of dependency (intermediate J_0), functioning and dysfunctional states of the system coexist.
- In systems with finite N, a functioning state can spontaneously switch to the dysfunctional state (without an apparent 'big' perturbation.)
- Results qualitatively unchanged for heterogeneous networks (not all-to-all interactions, heterogeneous levels of reliability, heterogeneous mutual dependency)
- Similar methods for credit risk ('fat tailed' loss distributions). Crises much more frequent than anticipated if interactions are neglected.
- Credit derivatives (CDS) can destabilise a system, if used to expand loan books.