Networking and Optimal Transport

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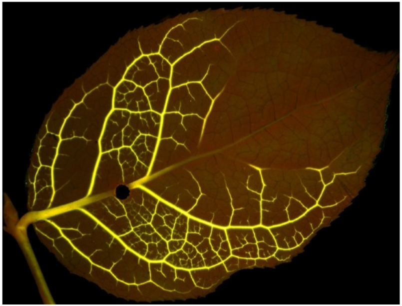
Overview

Overview of the presentation

- Optimal Transport on Network
- Leaf venation
- Interacting polymers
- The cavity approach

General framework

- **Distribution Networks:**

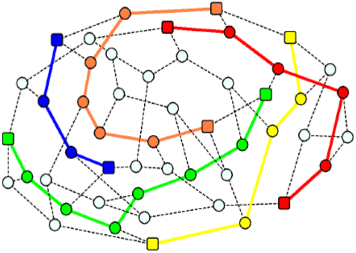


$G = \{V, E\}$: network
Supply of resource: Electricity, water, oxygen...
Nodes can be Sources and sinks. Find the configuration of currents that minimize the dissipation in the network

Static Problems

General Framework

- **Communication networks**
  a, b nodes; messages (...or car) μ that go from a^μ to b^μ.



Find the paths that constitute the best compromize between total path length and network congestion

Static Problems

Distribution Network

- Currents are generated and absorbed on nodes ($a = 1, ..., N$)

  $i_a > 0$ Source ;
  $i_a < 0$ Sink

- They circulate through edges:
  I_{ab} current circulating on edge $(a \to b)$ (directed)

- Current conservation (Kirchoff law).

  $i_a = \sum_b I_{ab}$

Distribution Network

- Energy dissipations J

  Edges; characterized by
  "distance" d_{ab} ; intrinsic
  conductance $K_{ab} = k_{ab} d_{ab} = 1/R_{ab}$; they can be changed.

- $J = \sum_{(ab)} \frac{I_{ab}}{d_{ab} k_{ab}}$

- Cost to build up the network $C[\{k_{ab}, d_{ab}\}]$

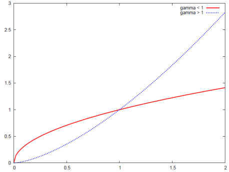
$$C[\{k_{ab}\}] = \sum_{ab} k_{ab}^{\gamma}$$

S. Bohn, M. Magnasco PRL 98, 088702 (2007)

Distribution Network

- Minimize dissipation for fixed construction cost

$$C[\{k_{ab}\}] = \sum_{ab} k_{ab}^{\gamma} = K$$



- $\gamma > 1$ (Convex cost) many small transmission lines
- $\gamma < 1$ (Concave cost) few big transmission lines

Result of optimization

- Triangular Lattice
- One source, $N-1$ sinks.
- Various values of γ.

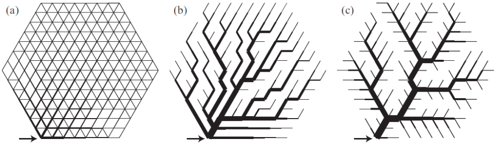
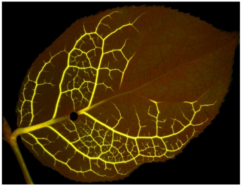


FIG. 2: Examples of the conductivity distributions. Results of the relaxation algorithm with different initial conditions: (a) $\gamma = 2.0$ and (b) $\gamma = 0.5$. (c) Result with optimized topology with $\gamma = 0.5$. the arrow indicated the localized inlet, the remaining nodes are outlets with constant i_k.

- Unique (smooth) minimum for $\gamma > 1$
- Local minima for $\gamma < 1$. Trees.

A Little theorem

If $C[\{k_{ab}\}]$ is concave the optimal transport network ia a Tree.



Loops emerge from other needs:

- Robustness of transport to damage
- Fluctuations in the network use.

Robustness of transport to damage

- Compute the power dissipated if P_{ab} if the edge (ab) is removed
- Consider as function to be optimized

$$R = \sum_{(ab)} P_{ab}$$

Katifori et al. PRL 2008

Fluctuating Network Usage

- Assume that i_a are fluctuating with probability $P(\mathbf{i})$
- Optimaze the *average* dissipation

$$J_{av} = \sum_{ab} \frac{\langle I_{ab}^2 \rangle}{K_{ab}}$$

Katifori et al. PRL 2010
Carlson PRL 2010

Loops & Hierarchy

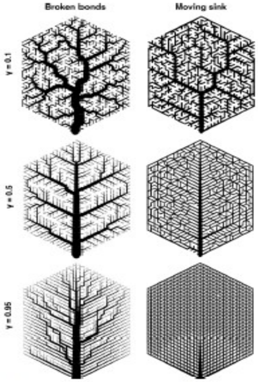
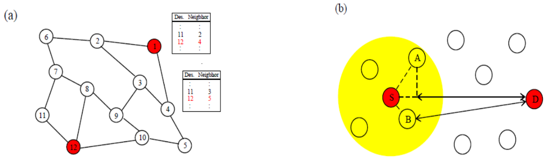


FIG. 2: Loops as a result of optimizing under damage to links (left column) and under a fluctuating load (right column). In all plots the vein thickness is proportional to $C^{(\gamma+1/2)/2}$.

Communication Networks

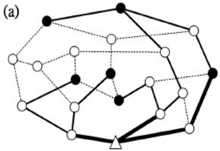
- **Networking:** routing, frequency allocation, information spreading, dynamic network allocation.
- Routing: Find the best route from *a* to *b*.



- Table based; Not dynamical ; insensitive to congestion
- Greedy: transmit to the neighbour close to destination

Frequency allocation to minimize interferences. Graph Coloring kind of problem.

Routing and the Physics of Polymers



- Neglect time: Static Paths on networks. M paths $M \sim N$.
  Chi Ho Yeung and David Saad PRL 2012
- Avoid Congestion : Interaction between paths.

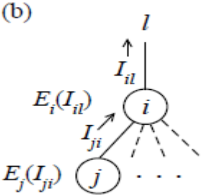
$$\sigma^{\mu}_{ab} = 0, \pm 1 \qquad I_{ab} = \sum_{\mu} \sigma^{\mu}_{ab} - \sigma^{\mu}_{ba}$$

$$E[\{I_{ab}\}] = \sum_{ab} I^{\alpha}_{ab} \qquad i^{\mu}_{a} = \sum_{b} \sigma^{\mu}_{ab}$$

$\alpha > 1$, minimize length, penalise congestion.
Polymers on networks with repulsive interaction.
At each node: A separate conservation law for each polymer.

Cavity approach

- At each node: A separate conservation law for each polymer.
- Defining Messages on Edges.
- Message $E_{a\to b}(\Sigma)$; $\Sigma = \{\sigma^1, ..., \sigma^M\}$



$$E_{a\to b}(\Sigma_{ab}) = I^{\alpha}_{ab} + \min_{\{I|\ \textit{Constraints}\ \}} \sum_{c\in\partial a-b} E_{c\to a}(\Sigma_{ca})$$

2^M messages at each link. Simplifications needed.

Cavity approach

- Multiple sender ; Single receiver.
- A unique conservation law at each node.

$$i_a = \sum_b I_{a,b}$$

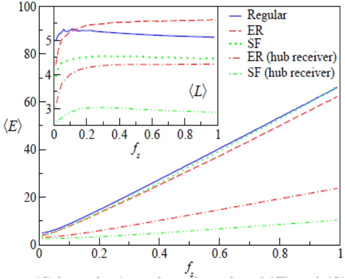
$$E_{a\to b}(I_{ab}) = I_{ab}^{\alpha} + \min_{\{I|\ \textit{Constraint}\ \}} \sum_{c\in\partial a-b} E_{c\to a}(I_{ca})$$

$|I| < M$. $\alpha > 1$ iterative procedure to compute the messages. Convex cost function.

Some results

Study of average Energy diss. and Length as a function of the fraction of senders f for different graph models.

- Erdos-Renyi, Random regular, Scale free



Some generalizations

- Non Convex cost functions
- Different kind of Receiver and Senders
- Effect of exclusion
- Free-flow and Jammed Phases.