Some aspects of statistical mechanics of disordered systems : Optimization

and communication (Silvio Franz)

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Outline

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Disordered systems

Many different physical systems in which disorder plays a crucial role :

- spin-glasses

  $$H = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \qquad \text{with some } J_{ij} > 0, \text{ others } < 0$$

  ordering at low-temperature in a frustrated,
  disordered groundstate

- disordered conductors

  $$H = -t \sum_{\langle ij \rangle} [a_i^+ a_j + a_j^+ a_i] + \sum_i \varepsilon_i a_i^+ a_i$$

  ε_i random, Anderson localization transition at large disorder

Disordered systems

- contact line of a fluid on a disordered substrate

[Moulinet, Guthmann, Rolley 02]

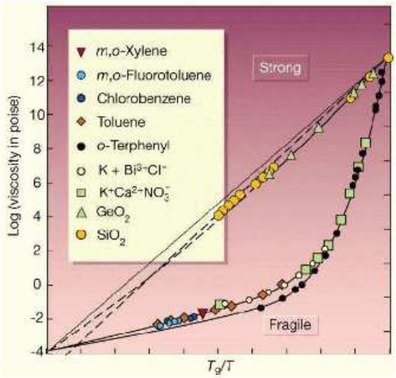
elastic energy + disordered pinning interaction

(also for vortices in supraconductors, domain walls in ferromagnets...)

studied via functional renormalization group

Disordered systems

- structural glasses



particles freezes in a disordered, amorphous, configuration

here the disorder is self-induced

Mean-field spin-glasses and optimization problems

Generically, mean-field theories provide a starting point for the physical understanding of a phenomenon

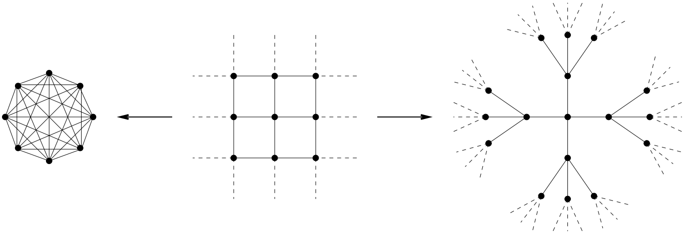
For spin-glasses : $H = -\sum_{i<j} J_{ij} \sigma_i \sigma_j$

J_{ij} i.i.d. Gaussians of mean 0, variance $\frac{1}{N}$ [Sherrington-Kirkpatrick, 75]

solution with the replica method [Parisi 80]

rigorous proof for the free-energy [Guerra-Toninelli, Talagrand 06]

Two kind of mean-field approximations :



Fully-connected (Curie-Weiss, Sherrington-Kirpatrick) vs
finite-connectivity (Bethe lattices) models

more realistic : connectivity remains finite in the thermodynamic limit

for instance Erdös-Rényi random graphs : among the $N(N-1)/2$ possible edges, keep each with probability α/N

average connectivity is α

Finite-connectivity mean-field spin glass : [Viana-Bray 85]
put random couplings on the edges of a random graph

Still mean-field, technically more involved than fully-connected models

- replica method [Monasson 98]
- cavity method [Mézard-Parisi 01]

Mean-field spin-glasses and optimization problems

Connection with combinatorial optimization (computer science) ?

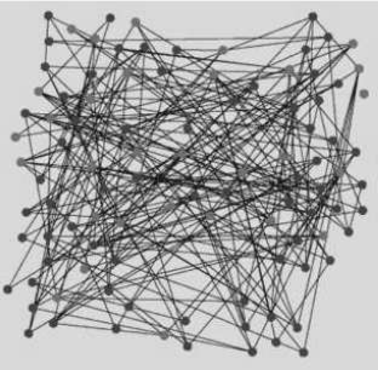
cost function $E(\underline{\sigma})$, on a discrete set of configurations
(for instance $\underline{\sigma} = (\sigma_1, \dots, \sigma_N)$, N boolean variables)

search for the minima of E

call E an energy (Hamiltonian), equivalent to low temperature statistical mechanics (variables $\leftrightarrow$ spins)

Mean-field spin-glasses and optimization problems

Example of the coloring problem : given a graph,

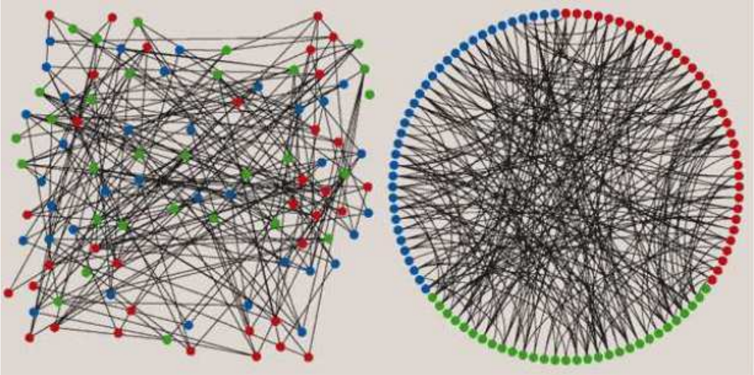


is it possible to color it with q colors without monochromatic edges ?

cost function = number of monochromatic edges = energy of a Potts antiferromagnet

in physics terms, is there a zero-energy groundstate ?

Mean-field spin-glasses and optimization problems



here the answer is yes for $q = 3$

but in general it is hard to answer (NP-complete for $q \geq 3$)

Mean-field spin-glasses and optimization problems

Very nice computational complexity theory

but the worst-case point of view might give too much importance to very rare very hard instances

typical complexity of a problem ?

studied with random ensembles of optimization problems

for example, coloring Erdös-Rényi random graphs

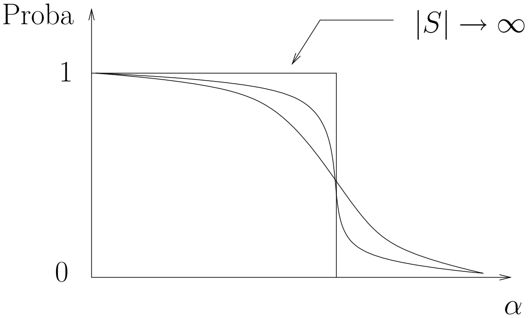
equivalent to mean-field spin-glasses

Outline

Phase transitions in random optimization problems

Random ensembles of optimization problems exhibit (several) phase transitions

Probability that a random graph is q-colorable :

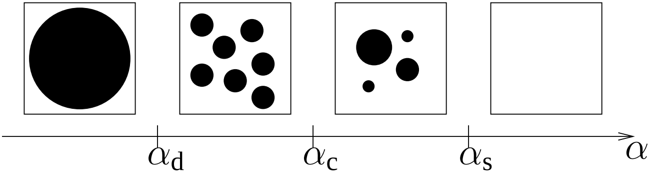


Sharp threshold (0/1 law) in the thermodynamic limit, at α_s

First discovered numerically in the 90's

Physical insight about (spin-)glasses helped to unveil other phase transitions for $\alpha < \alpha_{\rm s}$

Solutions (zero-energy groundstates) organized in clusters for $\alpha > \alpha_{\rm d}$:



exponentially many clusters for $\alpha \in [\alpha_{\rm d}, \alpha_{\rm c}]$

[Biroli-Monasson-Weigt 00]
[Mézard-Parisi-Zecchina 02]
[Mézard-Palassini-Rivoire 05]
[Krzakala-Montanari-Ricci-Tersenghi-GS-Zdeborova 07]

obtained via the (non-rigorous) replica/cavity methods

partial rigorous results, mainly at large *q* [Achlioptas, Coja-Oghlan 08] [Molloy 12]

Outline

1. Introduction
   - Disordered systems
   - Mean-field spin-glasses and optimization problems
2. Phase transitions in random optimization problems
3. The cavity method

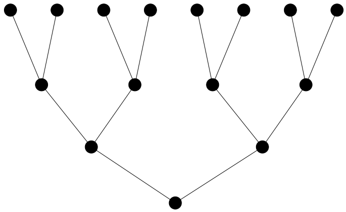
The cavity method

random graphs converge locally to trees

models on finite trees are simple

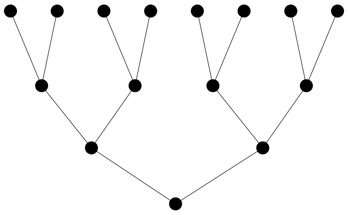
Simplest example : ferromagnet on a regular tree

$$H(\sigma_1,\dots,\sigma_N) = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j \ , \qquad Z = \sum_{\sigma_1=\pm1}\cdots\sum_{\sigma_N=\pm1} e^{-\beta H(\sigma_1,\dots,\sigma_N)}$$



The cavity method

$$H(\sigma_1, \dots, \sigma_N) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \ , \qquad Z = \sum_{\sigma_1 = \pm 1} \cdots \sum_{\sigma_N = \pm 1} e^{-\beta H(\sigma_1, \dots, \sigma_N)}$$

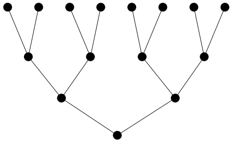


$Z_g(\sigma)$: partition function

- conditioned on the value of the root σ
- in a regular tree with g generations

The cavity method

$$Z_{g+1}(\sigma) = \sum_{\sigma_1,\dots,\sigma_{\tilde{c}}} Z_g(\sigma_1)\dots Z_g(\sigma_{\tilde{c}})\, e^{\beta J \sigma(\sigma_1+\dots+\sigma_{\tilde{c}})}$$



Normalized probability : $\eta_g(\sigma) = \frac{Z_g(\sigma)}{Z_g(+)+Z_g(-)} = \frac{e^{\beta h_g \sigma}}{2\cosh(\beta h_g)}$

Recursion on the effective magnetic field :

$$h_{g+1} = \frac{\tilde{c}}{\beta} \text{atanh}\left(\tanh(\beta J)\tanh(\beta h_g)\right)$$

Fixed point when $g \to \infty$: (infinitesimal field to break the symmetry)

- $h = 0$ at high temperature
- $h \neq 0$ at low temperature

True magnetic field can be recovered

with c instead of $\tilde{c}$ neighbors on the root

The cavity method

- Generalization to any model on a finite tree :

  solvable via exchange of “messages” between neighboring variables

- Generalization to models on random graphs :

  only locally tree-like, effect of the loops (boundary conditions)

  - for $\alpha < \alpha_{\rm d}$, Replica Symmetric phase, fast correlation decay
  - for $\alpha > \alpha_{\rm d}$, Replica Symmetry Breaking, correlated boundary conditions, computation of the number of clusters (pure states)

Method applicable to any model with an interaction graph converging locally to a tree, random constraint satisfaction problems, lattice glasses, properties of random graphs...