Investigating random laser through study of disordered systems.

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Motivation

Unlike conventional laser, random laser seeks amplification of electromagnetic waves by scattering of light among several scatterers placed randomly inside the lasing medium. A mean field approach is considered and the interaction among the light modes is studied. Where all the modes are connected with each other, the interaction among the competing modes depending on the mutual spatial overlap of their electromagnetic fields which is modified by a non linear susceptibilty is studied. Localization (due to disorder) together with the gain hence induces mode locked pulses in the spectrum.

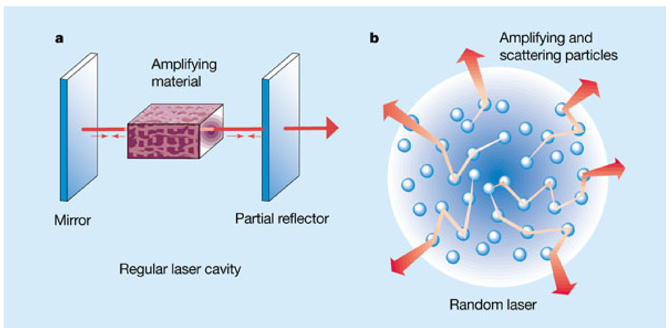

Figure 1: *Random laser v/s conventional laser*

Starting from the Haus master equation for mode locking lasers in disordered cavity:

$$\frac{\partial a}{\partial t} = (g - l + iD_n)a_n + (\gamma - i\delta)\sum_{\omega_i+\omega_l=\omega_j+\omega_k} a_j^* a_k a_l + \eta_n(t)$$

we reach the Hamiltonian with 2 body and 4 body terms together which reads as:

$$H = -\Re[\sum_{i<j} J_{ij}^{(2)} a_i a_j^* + \sum_{\omega_i+\omega_l=\omega_j+\omega_k} J_{ijkl}^{(4)} a_i a_j a_k^* a_l^*]$$

where the coupling with 4-body term J_{ijkl} governs the effective interaction among mode amplitudes:

$$J_{ijkl} = \int d^3r \chi^{(3)}_{\alpha\beta\gamma\delta}(\omega_i, \omega_j, \omega_k, \omega_l; r) E_i^{\alpha}(r) E_j^{\beta}(r) E_k^{\gamma}(r) E_l^{\delta}(r)$$

Coupling's amplitudes are quenched random variables and modes are required to satisfy the following **mode locking condition**:

$$\omega_i + \omega_l - \omega_j - \omega_k \leq \omega_0$$

ω_0 being the width of the peak in amplitude spectrum.
Also, the light mode amplitudes are governed by a global energetic constraint:

$$\sum_i a_i a_i^* = \mathcal{E}$$

Objective

4-ples spin glass model with a continuous variables is a mean field model for non linear random optical systems or random lasers which has been studied in [2] under the framework of replica method. But we need to reconsider the system since mode locking condition implies dilution in the graph compulsorily. In the previous case in fully connected system, all the modes are practically spatially extended to the whole system at frequency ω_0 but in more realistic random lasers dilution occurs because of the spatial localization structure of the modes and because of the mode locking transition. In this regard we need to study an equivalent system of 4 body spin glass model with continuous variables on a diluted or sparsely connected system.

Our task is to study coupling of modes in such a system using cavity method. To accomplish this task we first consider several studies already done in this regard on spin glass.

- cavity method on spin glass system with Ising variables in a fully connected model.
- XORSAT problem equivalent to spin glass system with Ising spins studied on a diluted graph.
- spin glass with continuous variables on a diluted graph.

Taking a step forward to these studies we aim to investigate in the problem of XORSAT with continuous variables on a sparsely connected graph.

Step1 : Ising spin fully connected graph

Cavity method on Bethe lattice Ising spin glass model in a fully connected system. It is in principle a formalism equivalent to replica method. It works on random graphs under Bethe approximation. Free energy can be calculated using the free energy shifts due to removal or addition of a spin in the graph. Observables are calculated if we add the cavity spin back to graph and study for all the messages passed to this spin via belief propagation.
Hamiltonian for a system of N Ising spins interacting with random couplings[3]:

$$H = -\sum_{<ij>} J_{ij}\sigma_i\sigma_j$$

The cavity field which is propagated along the tree:

$$u(J, h) = \frac{1}{\beta} atanh[\tanh\beta J \tanh\beta h]$$

More precisely if spin i is removed and except one(say j) all the neigbouring spins are reconnected then recursion relation for cavity fields reads:

$$u_{i\to j}(\sigma_i) = \frac{1}{z_{i\to j}}\prod_{k\epsilon\partial i\backslash j}(\sum_{\sigma_k} u_{k\to i}(\sigma_k) J_{ik}(\sigma_i, \sigma_k)),$$
$$u_i(\sigma_i) = \frac{1}{z_i}\prod_{j\epsilon\partial i}(\sum_{\sigma_j} u_{j\to i}(\sigma_j) J_{ik}(\sigma_i, \sigma_j))$$

Free energy will have both site and bond contribution:

$$F = -k\sum_i F_i^{(1)} + \sum_{<ij>} F_{<ij>}^{(2)}$$

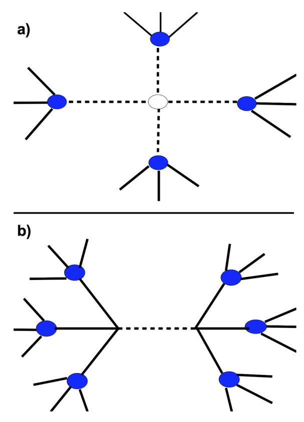

Figure 2: *site and link contribution*

The system can be solved by cavity method using population dynamics at the level equivalent to replica symmetry and replica symmetry breaking case where approximation for presence of several equivalent states is considered. Hence various observables like free energy, magnetisation, overlap order parameter be calculated.

Step 2: XORSAT Ising spin diluted graph

XOR-SAT is a constraint satisfaction problem to solve a system of linear equation which involves finding a vector $\vec{x}$ of boolean variables satisfying the linear equations $A\vec{x} = \vec{b}(mod2)$. When mapped into spin glass problem then $\sigma_i = (-1)^{x_i}$ and $J_i = (-1)^{b_i}$ resulting in Hamiltonian which tells us about the number of violated constraints[1]:

$$H(\sigma) = \sum_{a=1}^{M} \frac{(1 - J_a\prod_{j\epsilon\partial a}\sigma_j)}{2}$$

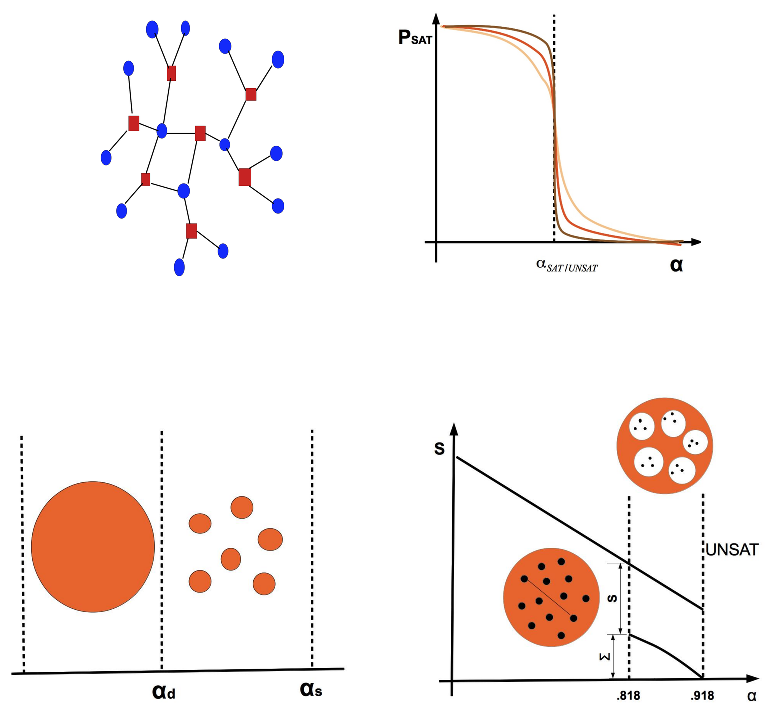

Figure 3: *from top clockwise :XORSAT factor graph, SAT-UNSAT transition,complexity graph and easy-hard SAT phase*

Solutions for such a system can be obtained and it can be checked whether the system is SAT or UNSAT. Alternatively, such a system can be solved using cavity method under certain assumptions.

Step 3: continuous spin diluted graph

Finite connectivity is more realistic than fully connected and same argument applies for vector spins over Ising spins. **Finitely connected spins** was investigated upon where vector spins live on a sphere $(\sigma_i\epsilon S_{d-1})$ with random matrix interactions[4].

$$H(\{\sigma\}) = -J\sum_{i<j} c_{ij}\sigma_i \cdot U_{ij}\sigma_j + \sum_i V(\sigma_i)$$

where $U_{ij}$are unitary matrices representing rotations in $\mathbb{R}^d$ which are drawn randomly and independently from random matrix ensemble $P(U)$ and $P(U) = P(U^{\dagger})$ and $V(\sigma_i)$ is the onsite potential and quenched disorder as:

$$Prob(c_{ij}) = \frac{c}{N}\delta_{c_{ij},1} + (1 - \frac{c}{N})\delta_{c_{ij},0}$$

Where this partition function can be used to find free energy which takes both site and bond contribution as in previous case. Working in the one pure-state approximation phase transitions can be found using bifurcation analysis on a suitable graph(say Poissonian graph). Order parameter is a functional and hence functional moment expansion is required to find phase transitions. Population dynamics in this case requires iterating functional which is numerically challenging to perform. Also, numerical simulations requires to generate suitable random matrices.

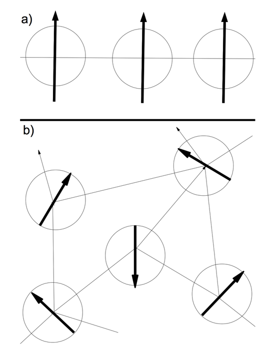

Figure 4: *Ising v/s chiral interaction*

For XY spins $\sigma = (\cos(\phi), \sin(\phi))$ with $\phi\epsilon[0, 2\pi)$ Probability distribution $P_i(\sigma_i)$ can be parameterized with anglle ϕ and so the rotations in the plane. Hence, we can write similar equation for order parameter and solve for it.

Coming to our point: XORSAT continous spin diluted graph

The basic Hamiltonian of N dimensional angular variables $\phi\epsilon[0, 2\pi]$ is given by :

$$H_J[\phi] = \sum_{i_1<i_2, i_3<i_4, i_1<i_3}^{1,N} J_i cos(\phi_{i_1} + \phi_{i_2} - \phi_{i_3} - \phi_{i_4})$$

where $cos(\phi_{i_1} + \phi_{i_2} - \phi_{i_3} - \phi_{i_4})$is the **state of the art** for our investigation where the phase $\phi_n = arg(a_n)$ with $A_n = |a_n|$ being the mode amplitude. We play on the ground of XOR-SAT problem considering state of the art for the quadruplets on a sparsely connected graph in order to investigate coupling of modes in a random laser medium. In a linear system of equations with 4 non vanishing entries in each row and with quadruple coupling $J_{ijkl}^{(4)}$ same as in the first section , a possible Hamiltonian be written and cavity method be done in order to investigate in the system.

References

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