Heterogeneity, interactions and credit risk

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Abstract

We present numerical results of a credit-risk model capturing effects of economic interactions on a firm's default probability, where an economy is represented by a graph undergoing a cascading contagion process over a finite time scale. We focus on scale-free networks and investigate the heterogeneity-induced departure from the Erdos-Renyi random graph situation. While Poissonian statistics induce largely gaussian behaviours in the distribution of defaults, scale-free networks exhibit decidedly non-gaussian spread. And while heterogeneous mean-field theory agrees with simulations in the weakly-interacting limit, stronger interactions show great sensitivity to finite-size effects.

Motivation

- Interest in the susceptibility to contagion of financial markets, especially in the wake of the Great Recession
- Heterogeneity in degree distribution is present in many real-life networks
- Analytic results are available for the Erdos-Renyi diluted network in the infinite connectivity limit, but not in the heterogeneous case or for low connectivity.

Setup

- N firms i, characterized by their wealth $W_{i,t}$ and status $n_{i,t} \in \{0, 1\}$
- Each node is connected to k_i other nodes j ($c_{ij} = 1$), and each edge (i, j) is given a weight J_{ij},

$$P(J_{ij}) \propto \mathcal{N}\left(\frac{J_0}{\langle k \rangle}, \frac{J}{\sqrt{\langle k \rangle}}\right)$$

- $W_{i,t} = \vartheta_i - \sum_j c_{ij} J_{ij} n_{j,t} - \eta_{i,t}$
- J_{ij} : loss incurred by i through default of firm j, $\eta_{i,t}$: intrinsic noise
- All firms are initially healthy ($n_{i,0} = 0$), a firm default when its wealth falls below 0 :

$$n_{i,t+1} = n_{i,t} + \left(\frac{1 - n_{i,t}}{2}\right) \Theta\left(\sum_j c_{ij} J_{ij} n_{j,t} + \eta_{i,t} - \vartheta_i\right)$$

- for convenience, we use $N = 500$ to keep the simulation reasonably short
- we are interested in the default distribution $P(\langle n \rangle)$ and the loss distribution, $P(\langle L \rangle)$

Theory

- We can use De Dominicis-type field theory to reach an analytic expression for the mean defaulted fraction,

$$\langle n_t \rangle = \lim_{\psi_{i,t} \to 0} \partial_{\psi_{i,t}} \overline{\left\langle \exp\left(\frac{1}{N} \sum_i n_{i,t}\, \psi_{i,t}\right)\right\rangle}$$

- We consider two cases :
  – Poisson statistics (Erdos-Renyi random graph), $k \propto \frac{c^k}{k!}$
  – Power-law statistics (scale-free networks), $k \propto k^{-\gamma}$, $\gamma = 3$
- The dynamics reduce to a single-site dynamics with self-consistency constraints
- In the Poisson case with $c \to \infty, N \to \infty$, the resulting equations can be solved numerically (Hatchett & Kuehn 2006)
- In the general and scale-free case the resulting equations are difficult to solve, hence we turn to numerical simulation and heterogeneous mean-field theory.

Numerical simulations

- We simulate the dynamics for $N = 500$, aiming for a "realistic" mean default rate of 3%, and obtain the defaulted fraction distribution $P(n_t)$ and the loss distribution $P(L_t)$ in various configurations of interaction strength and degree distribution.
- The scale-free networks exhibit a strong dependence on the lower cut-off of the degree distribution, and an enhanced default rate compared to Eros-Renyi random graphs
- Both default distribution and losses distribution are much wider than the Erdos-Renyi random graphs
- With stronger interactions (higher default rate), power-law simulations additionally become very sensitive to system size, and recovering the infinite-network size requires too much computing power for brute-force computation

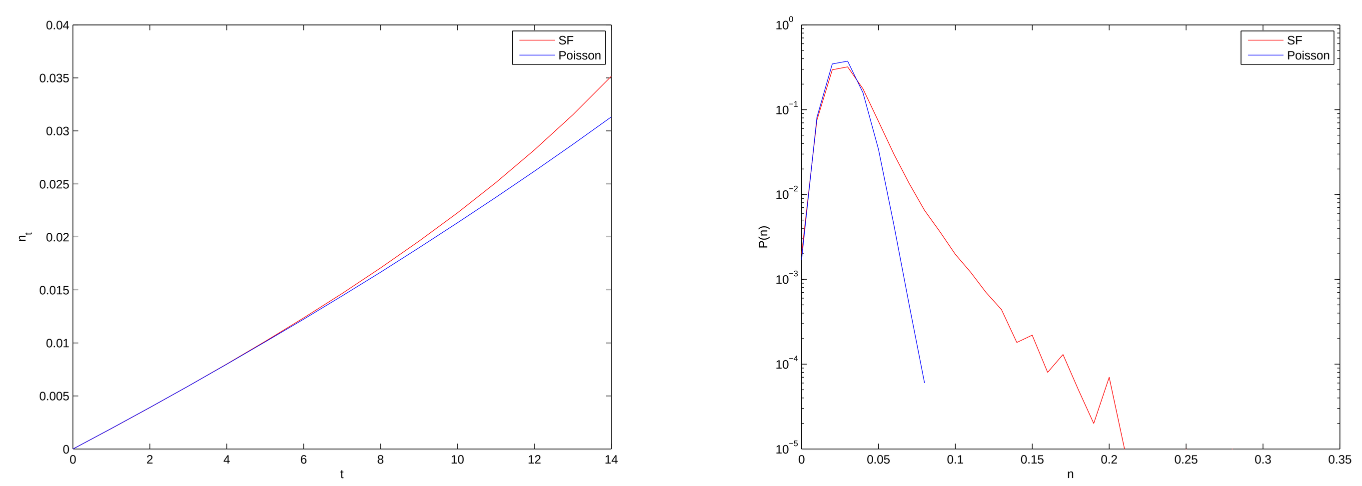

Figure 1: Comparison of scale-free networks with fixed-connectivity random graphs.
Left : mean defaulted fraction as a function of time, $\langle k \rangle = 5$, $J_0 = 0.7$, $J = \frac{\sqrt{J_0}}{2}$
Right : defaulted fraction distribution, same parameters

Heterogeneous mean-field approximation

- We assume that $\overline{n}_{i,t} \simeq n_t(k_i) = \frac{1}{N\,p(k_i)} \sum_j \delta_{k_i,k_j} \overline{n}_{j,t}$, i.e. a node follows its degree-class average.
- The result agrees very well with simulations in the Poisson case and in the weakly-interacting scale-free networks
- With scale-free networks with stronger interactions, the defaulted fraction n_t is strongly dependent on the upper cut-off of the degree distribution, mirroring the system-size dependence of the full simulation

Summary

- We have obtained default fraction and losses distributions for an idealized economy in a realistic network topology
- Interactions in the heterogeneous network induce a non-gaussian, fat-tailed default distribution
- In the weakly interacting case, it is suitably modelled by heterogeneous mean-field dynamics
- More work is needed in the heterogeneous case to take into account the finite-size effects