Mean field spin glasses treated with PDE techniques

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Abstract

We solve the **SK model** both at the replica symmetric and at the 1–RSB level, obtaining the correct expression for the free energy via an analogy to a **Fourier and Burger equation**, whose **shock wave** develops exactly at critical noise level (triggering the phase transition).

This approach, beyond acting as a new alternative method for tackling the complexity of spin glasses, let us to obtain a **new class of polynomial identities** (namely of **Aizenman-Contucci type**), whose interest lies in understanding, via the recent Panchenko breakthroughs, how to force the overlap organization to the ultrametric tree predicted by Parisi.

Model

Once introduced N Ising spins $\sigma_i \pm 1$, the **Hamiltonian of the SK model** is given by

$$H_N(\sigma;J) = -\frac{1}{\sqrt{N}}\sum_{(i,j)} J_{ij}\sigma_i\sigma_j$$

where the quenched disorder in the couplings is given by the $N(N-1)/2$ independent and identical distributed random variable J_{ij}, whose distribution is $\mathcal{N}[0,1]$

We are interested in an explicit expression for **the (quenched) free energy** $f(\beta)$ (or the **mathematical pressure** $\alpha(\beta)$) defined as

$$\alpha(\beta) = -\beta f(\beta) = \lim_{N\to\infty}\alpha_N(\beta) = -\lim_{N\to\infty}\beta f_N(\beta) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\ln Z_N(\beta)$$

RS scheme through an Interpolating Technique

We introduce two fictitious variables t, x which can be thought of as space and time coordinates, by which we write the **Guerra's interpolating function** as

$$\alpha_N(t,x) = \frac{1}{N}\mathbb{E}\ln\sum_\sigma \exp\left(\sqrt{\frac{t}{N}}\sum_{i<j}J_{ij}\sigma_i\sigma_j + \sqrt{x}\sum_i J_i^1\sigma_i\right)$$

where $\mathbb{E}$ is the average over the quenched couplings and the pressure is recovered whenever evaluating $\alpha(t,x)$ at $t=\beta^2, x=0$

We introduce the **Guerra's action** through a linear transformation in the (t,x) plan as

$$S_N(t,x) = 2\alpha_N(t,x) - x - t/2 \qquad (1)$$

which satisfies the following **Burgers equation**:

$$\partial_t S_N(t,x) + \frac{1}{2}\left(\partial_x S_N(t,x)\right)^2 - \frac{1}{2N}\partial^2_{x^2}S_N(t,x) = 0$$
$$= -\frac{1}{2}\left(\langle q_{12}^2\rangle - \langle q_{12}\rangle^2\right) + \lim_{N\to\infty}\left(\langle q_{12}^2\rangle - 4\langle q_{12}q_{23}\rangle + 3\langle q_{12}q_{34}\rangle\right) = 0$$

⇩ $=0$ ⇩ $=0$

satisfied in RS ansatz — it gives us the **Aizenman-Contucci identities**

Both satisfied in the thermodynamic limit

Fourier Formalism

We can map the latter equation to a **Fourier equation** via a **Cole-Hopf transformation**

$$\Psi_N(t,x) = \exp\left(-NS_N(t,x)\right)$$

by which it is straightforward to check that $\Psi_N(t,x)$ obeys the following **Fourier equation**

$$\frac{\partial\Psi_N(t,x)}{\partial t} - \frac{1}{2N}\frac{\partial^2\Psi_N(t,x)}{\partial x^2} = 0$$

which can be solved, in the Fourier space, trough Green propagator and the Convolution Theorem. Using the above Cole-Hopf transformation we can obtaining the following solution for the Guerra's action

$$S_N(t,x) = -\frac{1}{N}\ln\sqrt{\frac{N}{2\pi t}}\int dy\; e^{-N\left((x-y)^2/(2t) - \ln 2 - \ln\cosh(y)\right)} \qquad (2)$$

Applying the steepest descent method we can get the solution for the action in the limit $N\to\infty$ The extremization condition also gives

$$x = \hat{y} - t\int d\mu(z)\tanh^2(\sqrt{\hat{y}}z) = \hat{y} + t\,u(t,x)$$

Statistical mechanics is recovered at $x=0$ and $t=\beta^2$ at which the previous formula becomes:

$$\hat{y}_0 = \beta^2\,\bar{q} = \beta^2\int d\mu(z)\tanh^2(\beta\sqrt{\bar{q}}z) \longrightarrow$$

we recover the **self-consistent equation** for the order parameter

Then applying the steepest descent method in (2) and using the relation (1) between the action and the pressure, it is finally possible to obtain the **RS solution for the pressure:**

$$\alpha(\beta) = \ln 2 + \int d\mu(z)\ln\cosh(\sqrt{\bar{q}}\,\beta z) + \frac{\beta^2}{4}(1-\bar{q})^2$$

Shock waves coincide with phase transitions in statistical mechanics

Reminding the equation of motion $x = y + u(0,t)\,t$, from **mass conservation** we get

$$\rho(x) = \rho(y)\left(\frac{dx}{dy}\right)^{-1} = \frac{\rho(\bar{y})}{1+\partial_y u(0,y)|_{\bar{y}}t} = \frac{\rho(0)}{1-t}$$

which **diverges at the shock time** $t=\beta^2=1$, i.e. the critical noise level at which the phase transition occurs in statistical mechanics. Above $\bar{y}$ is the value which minimizes

$$\frac{dx}{dy} = 1 + t\,\partial_y u(0,y) = 1 - t\frac{1}{\sqrt{y}}\int d\mu(z)z\frac{\tanh(\sqrt{y}\,z)}{\cosh^2(\sqrt{y}z)}$$

1-RSB scheme through an Interpolating Technique

We go beyond the RS scenario and investigate the broken replica phase by merging the PDE approach with the broken replica interpolation scheme. The result will be a mapping between the SK free energy with 1-RSB step and a 2-dimensional diffusion equation. Let us introduce the **interpolating partition function**:

$$Z_N(t,x_1,x_2) = \sum_\sigma \exp\left(\sqrt{t}H_N(\sigma;J) + \sqrt{x_1}\sum_i^N J_i^1\sigma_i + \sqrt{x_2 - x_1}\sum_i^N J_i^2\sigma_i\right)$$

At the 1-RSB level we think at $m\in[0,1]$ as the expected parameter of the 1–RSB overlap's distribution and we assume its shape to be the weighted sum of two delta functions

$$P(q) = m\delta(q-\bar{q}_1) + (1-m)\delta(q-\bar{q}_2)$$

By introducing recursively the "partially averaged" partition functions as $Z_2 \equiv Z_N,\; Z_1^m = \mathbb{E}_2(Z_2^m)$, we can express the **1–RSB SK free-energy** (or **pressure** strictly speaking) in the space (t,x_1,x_2) as

$$\alpha(t,x_1,x_2) = \lim_{N\to\infty}\alpha_N(t,x_1,x_2) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\mathbb{E}_1\log Z_1(t,x_1,x_2) = \lim_{N\to\infty}\frac{1}{Nm}\mathbb{E}\mathbb{E}_1\log\mathbb{E}_2 Z_N^m(t,x_1,x_2)$$

If we now define the **Guerra's action for the 1-RSB scheme**

$$S_N(t,x_1,x_2) = 2\alpha_N(t,x_1,x_2) - x_2 - t/2, \qquad (3)$$

computing the streaming respect to x and t, we can immediately verify that the action satisfies:

$$\partial_t S_N + \frac{1}{2m}\left(\partial_{x_1}S_N\right)^2 + \frac{1}{2(1-m)}\left(\partial_{x_2}S_N\right)^2 = -\frac{1}{2}\left[m\left(\langle q_{12}^2\rangle_1 - \langle q_{12}\rangle_1^2\right) + (1-m)\left(\langle q_{12}^2\rangle_2 - \langle q_{12}\rangle_2^2\right)\right] \qquad (4)$$

where $\langle.\rangle_1 = \mathbb{E}[\Omega_1(.)]$ and $\langle.\rangle_2 = \mathbb{E}[f_2\Omega_2(.)]$ and the functions f are the following weights $f_1 = 1,\quad f_2 = \frac{Z_2^m}{\mathbb{E}_2 Z_2^m}$, while the standard product state remains $\Omega_a(.) = \omega_a(.)\times\ldots\times\omega_a(.)$ with the extended states defined as $\omega_1(.) = \mathbb{E}_2[f_2\omega_2(.)]$ and $\omega_2(.) = \omega_N(.)$

In the 1–RSB approximation we can neglect the r.h.s. of the equation (4) when $N\to\infty$; moreover, in analogy with the RS case, if we add a viscous term (with vanishing viscous coefficient) we can write down a **Burger equation** for $S_N(t,x_1,x_2)$ (having the same limiting solution) as

$$\partial_t S_N + \frac{1}{2m}\left(\partial_{x_1}S_N\right)^2 + \frac{1}{2(1-m)}\left(\partial_{x_2}S_N\right)^2 = \frac{1}{2Nm}\partial^2_{x_1^2}S_N + \frac{1}{2N(1-m)}\partial^2_{x_2^2}S_N \qquad (5)$$

which can be solved mapping it into a **Fourier equation** via a **Cole-Hopf tranform** $\Psi_N(t,x_1,x_2) = \exp\left(-NS_N(t,x_1,x_2)\right)$ by which it is straightforward to check that $\Psi_N(t,x_1,x_2)$ obeys

$$\partial_t\Psi_N(t,x_1,x_2) - \frac{1}{2Nm}\partial^2_{x_1^2}\Psi_N(t,x_1,x_2) - \frac{1}{2N(1-m)}\partial^2_{x_2^2}\Psi_N(t,x_1,x_2) = 0$$

Finally we can obtain the solution in the original space by the Convolution Theorem and come back to the action through the Cole-Hopf tranform:

$$S_N(t,x_1,x_2) = -\frac{1}{N}\log\Psi_N(t,x_1,x_2) = -\frac{1}{N}\log\int dy_1\int dy_2 e^{-N\left(S_0(y_1,y_2) + \frac{m}{2t}(x_1-y_1)^2 + \frac{(1-m)}{2t}(x_2-y_2)^2\right)}$$

Similarly to the RS case, in the thermodynamic limit the saddle point method gives also two equations of motion:

$$\begin{aligned}\hat{y}_1(t,x_1,x_2) &= x_1 + \langle q_{12}(x_1,x_2,t)\rangle_1 t = \beta^2(\langle q_{12}\rangle_1 = \beta^2\bar{q}_1\\ \hat{y}_2(t,x_1,x_2) &= x_2 + \langle q_{12}(x_1,x_2,t)\rangle_2 t = \beta^2\langle q_{12}\rangle_2 = \beta^2\bar{q}_2\end{aligned} \qquad (6)$$

where the last equalities hold when statistical mechanics is recovered, i.e. $x=0$ and $t=\beta^2$ The **self-consistent equations** are found out by combining the latter equations with the streaming of the action respect to x_1 and x_2. The **pressure** can be evaluated by the saddle point method and definition (3):

$$\begin{aligned}\alpha(\beta) &= \ln 2 + \frac{\beta^2}{4}\left(m\bar{q}_1^2 + (1-m)\bar{q}_2^2 - 2\bar{q}_2 + 1\right)\\ &+ \frac{1}{m}\int d\mu(z_1)\ln\int d\mu(z_2)\cosh^m\left(\beta\left(\sqrt{\bar{q}_1}z_1 + \sqrt{\bar{q}_2-\bar{q}_1}z_2\right)\right)\end{aligned}$$

where we have used the latter equality in (6) to recover the statistical mechanics.

New Overlap Constraints

The r.h.s of the diffusive equation (5) can be evaluated explicitly computing the streaming when evaluating the extremum via the steepest descent method. The results are related to the overlaps as follows:

$$-\frac{1}{Nm}\partial^2_{x_1^2}S = \frac{1}{N}\partial_{x_1}\langle q_{12}\rangle_1 \overset{N\to\infty}{\to} 0 \qquad \frac{1}{N(m-1)}\partial^2_{x_2^2}S = \frac{1}{N}\partial_{x_2}\langle q_{12}\rangle_2 \overset{N\to\infty}{\to} 0$$

If we now introduce the state $\tilde{\Omega}_1 = \mathbb{E}_2[f_2\Omega(.)]$ and define the composite states as $\langle.\rangle_{\tilde{\Omega}_1\times\Omega_1} = \mathbb{E}[(\tilde{\Omega}_1\times\Omega_1)(.)]$, after some computations, it possible to obtain an explicit expression for the above streaming which enforce the following **constraints for the overlaps:**

$$\begin{aligned}0 &= \langle q_{12}^2\rangle_2 + 2(m-2)\langle q_{12}q_{23}\rangle_2 + \frac{1}{2}(m-2)(m-3)\langle q_{12}q_{34}\rangle_2 + \frac{1}{2}m(1-m)\langle q_{12}q_{34}\rangle_{\Omega_1\times\tilde{\Omega}_1}\\ 0 &= \langle q_{12}^2\rangle_1 - 4m\langle q_{12}q_{23}\rangle_1 + 3m^2\langle q_{12}q_{34}\rangle_1 + 2(m-1)\langle q_{12}q_{13}\rangle_{\Omega_1\times\tilde{\Omega}_1}\\ &+ 4m(1-m)\langle q_{13}q_{24}\rangle_{\Omega_1\times\tilde{\Omega}_1} + (1-m)^2\langle q_{13}q_{24}\rangle_{\tilde{\Omega}_1\times\tilde{\Omega}_1} + 2m(m-1)\langle q_{12}q_{34}\rangle_{\Omega_1\times\tilde{\Omega}_1}\end{aligned}$$

References and contacts

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